Von Karman equation of thin elastic plate by the ANM-MFS algorithm

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Abstract

In this work, the association of the Asymptotic Numerical Method (ANM) with the Method of Fundamental Solutions (MFS) to solve the Von Karman equation is investigated. The Von Karman equation introduced a system of two fourth order elliptic nonlinear partial differential equations which can be used to describe the large deflections and stresses produced in a thin elastic plate subjected to external loads. The MFS is one of the most developed meshless methods and is probably the most used Trefftz method. Thanks to the Taylor series development, the system of two fourth order elliptic nonlinear partial differential equations are transformed into a succession of a system of two linear bi-harmonic equations. Knowing that the fundamental solution is not always available, the MFS-RBF (Radial Basic Functions) method is combined with the Analog Equation Method (AEM) to solve these resulting system of two linear bi-harmonic equations and computes the nonlinear branch solutions. The efficiency of the method is verified through a numerical example.

Keywords : Method of Fundamental Solutions, Asymptotic Numerical Method, Von Karman Plate.

1 Introduction

Analysis of large deformations of square and rectangular plates is one of the most studied engineering problems in the structural community, with many engineering applications including in aircraft structures, shipbuilding, bridges and spaceships. The extension to large deformations was first provided by Von Karman in a seminal work [1], wherein the nonlinear terms are retained in the kinematic relationships to account for a significantly large deformation of the plate (w is comparable with plate thickness or larger but remains small with respect to other dimensions of the plate). This leads to a pair of coupled fourth-order nonlinear equations of transverse displacement and stress function.

The Von Karman equations for bending thin plates have been studied by various authors. The study of the numerical approximation of these equations by Finite Elements Method (FEM) has been undertaken by Miyoshi [2] and more recently by Brezzi [3] which demonstrates the convergence of Newton's method in the case of an isolated solution. The possibility of obtaining accurate numerical results without resorting to the mesh has been the aim of many researchers throughout the past two decades. The Method of Fundamental Solution (MFS)[4] is one of the most recently developed meshless methods that has attracted attention in recent years and is probably the most used Trefftz method. The main advantage of the MFS is a reduction of the number of unknowns and a simple theory implementation.

In the present work, a numerical technique for solving the Von Karman equations for thin plate is presented by combining ANM with MFS. The ANM method is a technique developed to compute the solution of nonlinear partial differential equations. It consists in transforming the nonlinear problem into a sequence of linear ones by expanding the unknowns in power series. As the series solution is limited by a radius of convergence, a continuation procedure allows one to obtain the whole solution branch in a step by step manner. The step length is computed a posteriori by exploiting the terms of these series. It has been successfully applied in nonlinear solid and fluid mechanics [5]. Nevertheless, several papers coupled ANM with a meshless discretization method [6, 7, 10, 11]. Recently, Tian et al. [8] have associated a method based on the ANM with Taylor Meshless Method (TMM) for post-buckling analysis within the Von Karman plate theory. Askour et al. [9] have associated MFS and ANM to solve 2D nonlinear elasticity problems. To our knowledge, the Von Karman plates theory has not been again investigated with MFS.

The aim of this paper is to offer a simple numerical technique (ANM with MFS) for the resolution of the Von Karman equations. The paper is organized as follows. The governing equations based on $w - F$ formulation are presented in Section 2. In section 3, we illustrate the ANM for solving the nonlinear equations, where the MFS is detailed in section 4. The efficiency of this method is demonstrated by numerical a example in section 5, followed by some concluding remarks in section 6.

2 Governing equations

The elastic large deflection response of a plate is governed by two fourth order elliptic nonlinear partial differential equations, which are named Von Karman equations [1]. One of them represents the equilibrium condition in the transverse direction and the other represents the compatibility condition of in-plane strains. The PDEs are as follows :

$$\left\{ \begin{array}{lcl} D\Delta^2 w - [F, w] & = & \lambda g \\ \\ \Delta^2 F + \frac{Eh}{2}[w, w] & = & 0 \end{array} \right. , \quad D = \frac{Eh^3}{12(1-\nu^2)} \tag{1}$$

In the above equation, w is the additional transverse displacement, F is the Airy stress function governing the in-plane stress resultants, g is the lateral pressure acting on the plate, λ is a scalar load parameter, D is the flexural rigidity, h is the plate thickness, E and ν are Young's modulus and Poisson's ratio. We use the notation Δ^2 for the bi-harmonic operator and the classical bracket operator $[f, q]$ is related to the Gaussian curvature and defined by :

$$[f, q] \quad = \quad \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 q}{\partial y^2} + \frac{\partial^2 f}{\partial y^2}\frac{\partial^2 q}{\partial x^2} - 2\frac{\partial^2 f}{\partial x \partial y}\frac{\partial^2 q}{\partial x \partial y} \tag{2}$$

Stress σ_x in the x direction, σ_y in the y direction and shear stress τ_{xy} in xy plane may be expressed as

$$\sigma_x = \frac{\partial^2 F}{\partial y^2}, \quad \sigma_y = \frac{\partial^2 F}{\partial x^2}, \quad \tau_{xy} = -\frac{\partial^2 F}{\partial x \partial y} \tag{3}$$

The boundary conditions for the transversal displacement are the same as those used for classical plate analysis, which is fulfilled in the following two cases.

The case of a clamped edge in the transverse direction is obtained if :

$$w \quad = \quad \frac{\partial w}{\partial n} \quad = \quad 0 \tag{4}$$

The case of a simply supported edge is obtained if :

$$\begin{cases} w & = \quad 0 \\ \\ M_{nn} & = \quad -D\left(\nu \nabla w + (1-\nu)\frac{\partial^2 w}{\partial n^2}\right) \quad = \quad 0 \end{cases} \tag{5}$$

where M_{nn} represents the normal bending moment.

The boundary conditions for the stress function F are obtained by assuming that the external edge of the plate is not subjected to in-plane forces which yields the following boundary conditions :

$$F \quad = \quad \frac{\partial F}{\partial n} \quad = \quad 0 \tag{6}$$

As a consequence, equation (6) is a sufficient condition to impose a free edge in an in-plane direction.

3 Asymptotic Numerical Method

The basic idea of ANM consists in searching the solution paths generated by (w, F, λ) of the nonlinear problem (1) in the form of a truncated Taylor series from a known and regular solution (w_0, F_0, λ_0) under the following form :

$$\begin{Bmatrix} w \\ \\ F \\ \\ \lambda \end{Bmatrix} = \begin{Bmatrix} w_0 \\ \\ F_0 \\ \\ \lambda_0 \end{Bmatrix} + \sum_{k=1}^{P} \begin{Bmatrix} w_k \\ \\ F_k \\ \\ \lambda_k \end{Bmatrix} \tag{7}$$

where P is the truncation order of the Taylor expansions and the path parameter "a" can be defined as :

$$a \quad = \quad < w - w_0, w_1 > + < F - F_0, F_1 > + (\lambda - \lambda_0)\lambda_1 \tag{8}$$

where $<\cdot\,,\cdot>$ is the Euclidian scalar product. The ANM relies on a numerical calculation of the Taylor series terms (7). Taking into account the series (7), the system (1, 8) becomes a set of linear problems by equating like powers of parameter "a". At the order $k = 1$, one gets a linear system satisfied by

(w_1, F_1, λ_1) and that is nothing but the tangent system :

$$\begin{cases} D\Delta^2 w_1 - [F_0, w_1] - [F_1, w_0] & = \lambda_1 g \\ \Delta^2 F_1 + Eh[w_0, w_1] & = 0 \\ < w_1, w_1 > + < F_1, F_1 > + \lambda_1^2 & = 1 \end{cases} \tag{9}$$

At the generic order k, the unknowns (w_k, F_k, λ_k) are solutions of another linear system involving the same linear operator as in (9) :

$$\begin{cases} D\Delta^2 w_k - [F_0, w_k] - [F_k, w_0] & = \lambda_k g + \sum_{r=1}^{k-1} [F_r, w_{k-r}] \\ \Delta^2 F_k + Eh[w_0, w_k] & = -\frac{Eh}{2} \sum_{r=1}^{k-1} [w_r, w_{k-r}] \\ < w_k, w_1 > + < F_k, F_1 > + \lambda_k \lambda_1 & = 0 \end{cases} \tag{10}$$

Finally, all vectors $(w_k,\ F_k)$ and the scalar parameters λ_k of series (7) can be determined by solving the system of equation (9) and (10) at each truncation order. We recall that all the linear problems (9) and (10) have the same tangent operator and different forms of right-hand sides. In fact, only one matrix decomposition in each ANM-step is needed. The validity range of the series (7) is limited by the convergence radius. To obtain the whole solution branch, a continuation technique is used. It consists in computing the step length of the solution automatically by the following formula [5] :

$$a_{max} = \left(\varepsilon \frac{\|w_1\|}{\|w_P\|} \right)^{\frac{1}{P-1}} \tag{11}$$

Here, ε is a small number and the norm $\| \cdot \|$ in (11) is chosen as the Euclidean norm. The solution $(w(a_{max}), F(a_{max}), \lambda(a_{max}))$ is a new starting solution for the following step. This technique allows us to compute a posteriori the step length of the solution, which is naturally adaptive and depends on the local nonlinearity of the considered problem.

4 Method of Fundamental Solutions

We propose to solve the resulting linear problems (9-10) by using Method of Fundamental Solutions (MFS). We approximate the solution to the bi-harmonic problem by a linear combination of fundamental solutions of both the Biharmonic operator and Laplace's operator and particular solution which based on Radial Basis Functions (RBF) :

$$\begin{cases} w(M_i) & = \sum_{j=1}^{N_s} G_1(M_i, Q_j) \alpha_j^w + G_2(M_i, Q_j) \beta_j^w + \sum_{j=1}^{N_s} \vartheta(M_i, M_j) \gamma_j^w \\ F(M_i) & = \sum_{j=1}^{N_s} G_1(M_i, Q_j) \alpha_j^F + G_2(M_i, Q_j) \beta_j^F + \sum_{j=1}^{N_s} \vartheta(M_i, M_j) \gamma_j^F \end{cases} \tag{12}$$

where Q_j and M_i are respectively the coordinates of N_s source points which are taking on the fictitious boundary and the coordinates of the N collocation points. G_1 represents the fundamental solutions of the bi-harmonic operator [12] :

$$G_1(M_i, Q_j) = \frac{\rho_{ij}}{8\pi} log(\rho_{ij}) \tag{13}$$

G_2 represents the fundamental solutions of the Laplace operator [12] :

$$G_2(M_i, Q_j) \;\; = \;\; \frac{1}{2\pi} log(\rho_{ij}) \tag{14}$$

where $\rho_{ij} = \sqrt{(x_i^M - x_j^Q)^2 + (y_i^M - y_j^Q)^2}$ and ϑ represents a particular solution of the bi-harmonic operator which is built from the Radial Basis Functions defined by [13] :

$$\Delta^2 \vartheta(r_{ij}) \;\; = \;\; \Phi(r_{ij}) \tag{15}$$

Effectiveness and accuracy of the interpolation depend on the choice of the approximating functions Φ. Global interpolation functions, such as Lagrange polynomials, Fourier sine and cosine series, RBFs of polynomial type and thin plate spline (TPS), multiquadric (MQ) functions may be used for this purpose. In this paper, the functions Φ in equation (15) are selected to be thin plate spline RBFs with respect to an Euclidian distance $r_{ij} = \sqrt{(x_i^M - x_j^M)^2 + (y_i^M - y_j^M)^2}$:

$$\Phi(r) \;\; = \;\; r^2 log(r) \tag{16}$$

Next the use of equations (15) and (16) gives the approximating particular solutions $\vartheta(r)$ by :

$$\vartheta(r) \;\; = \;\; 10^{-3} \frac{r^{10}}{120}(20 log(r) - 9) \tag{17}$$

After satisfying all equations of the resulting linear problems (9-10) and take into account the boundary conditions for w and F at collocation points, a system of equations is solved to determine the unknown coefficients $(\alpha_j^w, \beta_j^w, \gamma_j^w, \alpha_j^F, \beta_j^F, \gamma_j^F)$

5 Numerical results and discussions

5.1 Square plate subjected to lateral load

To validate our results, we considered a square plate with geometry $a = b = 2$ and thickness $h = 1$ subjected to an uniformly distributed lateral load. For generality, all quantities are made dimensionless ; the coordinates, the deflection, the load and the stress are represented by :

$$x = a\bar{x}, \quad y = b\bar{y}, \quad w = h\bar{w}, \quad g = \frac{Eh^4}{a^4}\bar{g}, \quad F = Eh^3\bar{F} \tag{18}$$

In this example, Young's modulus $E = 1$, Poisson's ratio $\nu = 0.3$ and the fictitious boundary is a circle of radius $R = 3$. We take $N = 197$ points distributed over the domain occupied by the plate and $N_s = 41$ represents the number of points on the fictitious boundary chosen equal to the number of the boundary points in the domain (see figure 1).

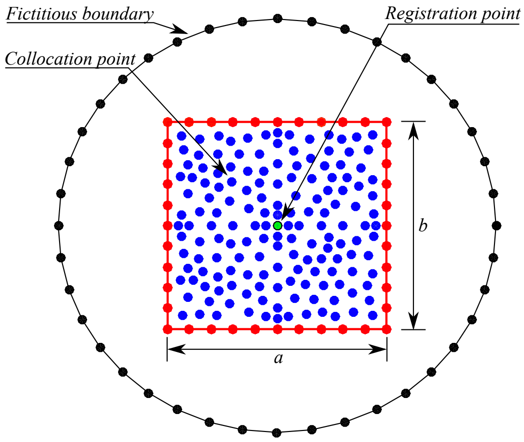
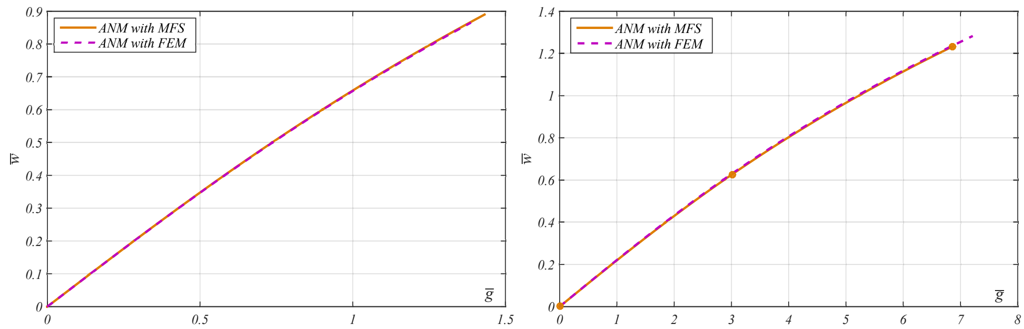


FIGURE 1 – Node distribution and fictitious boundary

The deflection-load evaluated at registration point for clamed and simply supported edge boundary condition is given in figure 2. There is no analytical solution available for this problem and therefore the MFS solution is compared with FEM solution. The parameters of the algorithm are the truncation order $N = 21$ and the tolerance parameter $\varepsilon = 10^{-8}$. For comparison purpose, this plate has also been analyzed via FEM using quadrilateral element with four nodes (Q4) and two degrees of freedom at each node ; 20×20 elements for the whole domain.



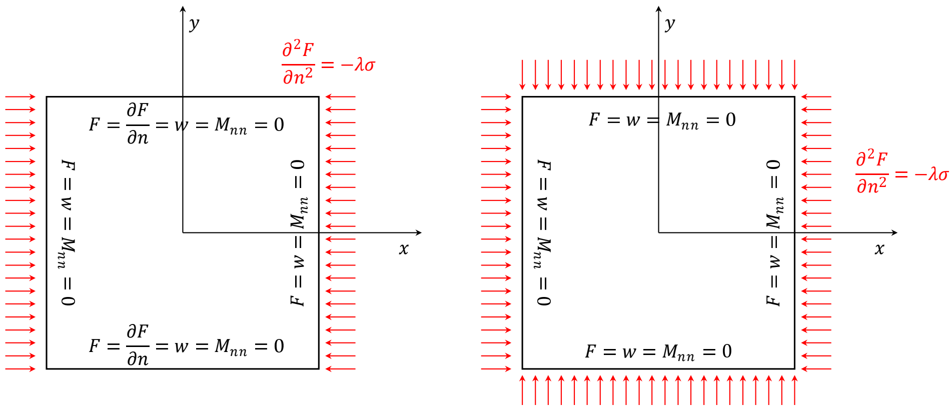
(a) Simply supported edge boundary condition (b) Clamped edge boundary condition

FIGURE 2 – Central deflection versus load

It is seen from figure 2 that the results of the present method are quite in accordance with those of the FEM.

5.2 Square plate under uniaxial and biaxial compression

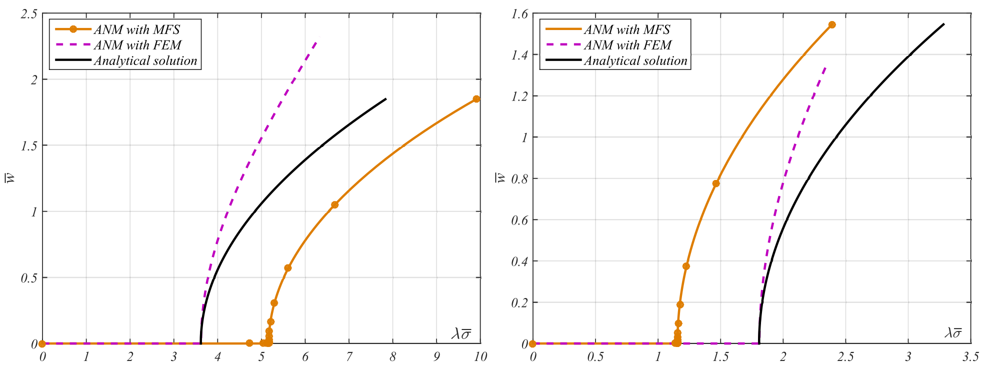
Let us consider a simply supported square plate under uniaxial and biaxial compression load (see figure 3). The most interesting phenomenon associated with this nonlinear situation is the appearance of "buckling". The thin plate may deflect out of its transversal direction when the compression load reaches a certain magnitude.



(a) Uniaxial compression load (b) Biaxial compression load

FIGURE 3 – Simply supported square plate subjected to compressive load

The data are written in dimensionless form ; Young's modulus $E = 1$, Poisson's ratio $\nu = 0.3$, thickness $h = 1$, width $a = 2$, the loading $\sigma = 1$. The dimensionless quantity of compression load is $\sigma = \frac{Eh}{a^2}\bar{\sigma}$. This buckling problem is solved by the proposed method and compared with finite element and analytical solution. In both method (MFS and FEM), we use a small perturbation to compute a bifurcating branch by path following, the perturbation will be an uniform transversal pressure. The response curve of the buckling problem is obtained by a truncation order $P = 21$ and a tolerance parameter $\varepsilon = 10^{-8}$. In figure 4 we present a solution branch obtained with 18 steps evaluated at registration point as the first example.



(a) Uniaxial compression load (b) Biaxial compression load

FIGURE 4 – Central deflection versus compressive load

In these figures, step accumulation is observed for $\lambda\bar{\sigma}$ close to 3.615 and 1.808 respectively for uniaxial and biaxial compression load obtained by (ANM with FEM) and is exactly the same as in the analytic solution, but our algorithm also detects the bifurcation point with a slight difference that can be linked mainly to the consideration of boundary conditions. After the steps accumulation closes the bifurcation point, the solution branch transits from the primary branch to the bifurcated one.

6 Conclusion

In this work, we have associated the Method of Fundamental Solutions with Asymptotic Numerical Method to solve the problem of large deflection of thin plates which is governed by a Von Karman equation. The proposed technique is successfully tested on a square plate subjected to the lateral load.

Indeed, the problem of square plate under uniaxial and biaxial compression have some defects to locate the critical load. This algorithm is currently in progress to study the thin plates buckling.

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