Numerical Study of unsteady Couette flow in discrete kinetic theory

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Abstract :

We investigate the unsteady flow of a discrete gas between two infinite moving and impermeable parallel plates. The first studies of the Couette flow problem in the scope of discrete kinetic theory were carried out in the steady case with discrete models having only one speed [1]. Only the behavior of the total density and the mean velocity has been examined. In the unsteady case, a work carried out with a class of four velocity models with the same modulus made it possible to study the evolution of total density, tangential and normal velocities as functions of time [2]. The objective of the present study with a ten velocity discrete model with two different speeds [3], is to study the transition from unsteady flow to steady flow by analyzing the behavior of the macroscopic variables of the flow (total density, kinetic temperature, normal and tangential velocities). The problem is solved numerically using the fractional step method. By varying the Knudsen number Kn, we pass from continuous flows to rarefied flows. The study therefore focus on finding and evaluating the effects of rarefaction highlighted by other methods of investigation and analyzing their evolution with time.

Keywords : Kinetic theory ; Discrete velocity models ; Unsteady Couette flow ; Transition ; Steady state.

1 Introduction

The problem of the flow of a gas and the transfer of energy between two infinite and mobile parallel plates, although it can be considered as a simple problem of gas dynamics, does not have exact solutions obtained by the resolution of the complete Boltzmann equation. When the flow is steady, the discrete kinetic theory allows to solve analytically this problem in certain cases and to make a qualitative study of the solutions. The first studies of the Couette flow problem in discrete kinetic theory were carried out with one speed discrete velocity models [1]. Since these models can not account for energetic phenomena, only the behavior of the total density and the mean velocity has been examined. Later works used discrete velocity models with several speeds and one has been able to solve analytically the problem of the flow of a discrete gas between two infinite parallel plates with the same temperature and opposite velocities [4], and numerically the problem of heat transfer between a gas and two infinite parallel plates at rest and having different temperatures [5]. But the particular boundary conditions adopted in these studies

do not permit to bring to the fore rarefaction phenomena found using other methods of resolution. By adopting the boundary conditions of diffuse reflection on infinite plates with arbitrary temperatures and velocities, it has been possible to highlight the phenomena of increase of internal energy in the flow as well as the existence of Knudsen layers in the vicinity of the plates inside which velocity slip and temperature jump occur [3]. Relatively, few studies concern unsteady Couette flow in discrete kinetic theory. The first work carried out with a four velocity discrete model with the same modulus made it possible to study the evolution of the total density, the tangential and normal velocities as a function of time [2].
In the sequel, using a ten velocity discrete model with two different speeds, we study the transition from unsteady flow to steady state of a gas initially at rest and suddenly set in motion by the displacement of a plate in contact with it. In section 2 we recall some basic results of discrete kinetic theory and describe the model $C1$. The physical problem is set and the numerical method of resolution is presented in section 3 and section 4. Finally in section 5 the results are discussed.

2 Description of the model

A discrete model of gas is a medium composed of particles whose velocities belong to a given discrete set of vectors. The theory is well know [6, 7] and a p velocity discrete model is a medium whose particles velocities belong to a given set of p vectors $\vec{u}_i, i \in \wedge = \{1, \cdots, p\}, p \in \mathbb{N}$. The number density of particles of velocity $\vec{u}_i$ at time t' and point $\vec{x'}$, is denoted $N_i(t', \vec{x'})$. Depending on the density of the medium, binary or high orders collisions occur. For sake of simplicity, only binary collisions are retained for this study. The Boltzmann equation is replaced by a system of coupled semi-linear partial differential equations which describe the evolution of the microscopic densities associated with each of the selected velocities. In absence of external force and when the molecular chaos is initially assumed, the balance equation for particles with velocity $\vec{u}_i$ is

$$\frac{\partial N_i}{\partial t'} + \vec{u}_i \cdot \nabla N_i = \frac{1}{2} \sum_{j,k,l} A_{ij}^{kl} \left(N_k N_l - N_i N_j \right), \quad i \in \wedge \tag{1}$$

where the coefficients A_{ij}^{kl} are the transition probabilities associated with the collision $(\vec{u}_i,\ \vec{u}_j) \longleftrightarrow (\vec{u}_k,\ \vec{u}_l)$.
The determination of the $N_i(t', \vec{x'})$, $i \in \wedge$, gives the microscopic description of the discrete model of gas. The macroscopic variables of a discrete model of gas are associated to the summational invariants of the model. The summational invariants [8] are quantities conserved through collisions. The summational invariants attached to the conservation of mass, momentum and energy are called physical invariants. In contrast to the classical kinetic theory of monoatomic gases, the geometric character of the set of the given velocities may allow other summational invariants called spurious invariants or cause the physical invariants to be linearly dependent [3, 9].

The total density N, the macroscopic velocity $\overrightarrow{U}$ and the total energy E are defined by [9] :

$$N = \sum_{i \in \wedge} N_i, \quad N\overrightarrow{U} = \sum_{i \in \wedge} N_i \vec{u}_i, \quad NE = \frac{1}{2} \sum_{i \in \wedge} N_i \vec{u}_i^2. \tag{2}$$

In the tridimensional physical space the kinetic temperature T is defined by

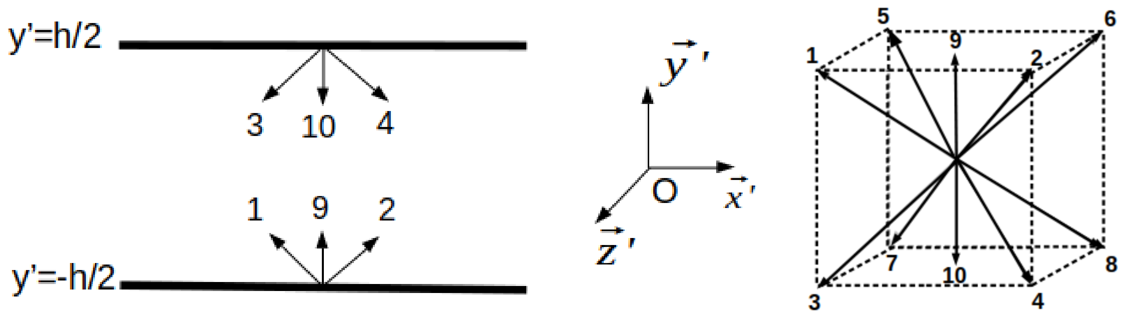
$$\frac{3k_BT}{2m} + \frac{\overrightarrow{U}^2}{2} = E \tag{3}$$

where k_B is the Boltzmann constant and m is the particle mass.

The thermodynamic equilibrium state of the discrete model of gas is called Maxwellian state. For a discrete velocity model having linearly independent physical invariants, the microscopic densities in a Maxwellian state associated with the macroscopic variables N, $\overrightarrow{U}(U, V, W)$ and E are [9] :

$$N_i \quad = \quad \exp(\lambda_0 + \vec{\lambda} \cdot \vec{u}_i + \lambda_4 \vec{u}_i^2), \quad \forall i \in \wedge \tag{4}$$

where the parameters λ_0, $\vec{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$ and λ_4 are uniquely determined by the implicit relations (2). The discrete model in consideration in this work, the ten velocity model $C1$, has only linearly independent physical invariants. The set of its velocities is $\{\vec{u}_i = (U_i, V_i, W_i) : i \in \wedge\}$ [9] where $\wedge = \{1, \cdots, 10\}$ and $\vec{u}_1 = c(-1, 1, 1)$, $\vec{u}_2 = c(1, 1, 1)$, $\vec{u}_3 = c(-1, -1, 1)$, $\vec{u}_4 = c(1, -1, 1)$, $\vec{u}_{9-j} = -\vec{u}_j$, $j \in \{1,\ 2,\ 3,\ 4\}$, $\vec{u}_9 = -\vec{u}_{10} = c(0, 1, 0)$, $c > 0$ is a characteristic speed of the phenomenon in consideration and will be given in the sequel.



(a) Geometry of the flow (b) Velocities of model $C1$

FIGURE 1

The model has two different speeds and therefore two kinds of collisions : collisions between particles having the same speed and collisions between particles of different speeds. Depending on the relative velocity of the colliding particles the transition probability of the collisions of particules having the same speed are either $A_{14}^{23} = cs\sqrt{2}$ or $A_{18}^{27} = cs\dfrac{\sqrt{3}}{2}$. The transition probability of the collision between particles with different speeds is $A_{19}^{310} = cs\dfrac{\sqrt{6}}{2}$.

The binary collisions of the model $C1$ are the following :

$$\begin{array}{ll}
(\vec{u}_1,\ \vec{u}_8) \longleftrightarrow (\vec{u}_2,\ \vec{u}_7) \longleftrightarrow (\vec{u}_3,\ \vec{u}_6) \longleftrightarrow (\vec{u}_4,\ \vec{u}_5), & \\
(\vec{u}_1,\ \vec{u}_4) \longleftrightarrow (\vec{u}_2,\ \vec{u}_3), & (\vec{u}_1,\ \vec{u}_6) \longleftrightarrow (\vec{u}_2,\ \vec{u}_5), \\
(\vec{u}_1,\ \vec{u}_7) \longleftrightarrow (\vec{u}_3,\ \vec{u}_5), & (\vec{u}_4,\ \vec{u}_6) \longleftrightarrow (\vec{u}_2,\ \vec{u}_8), \\
(\vec{u}_4,\ \vec{u}_7) \longleftrightarrow (\vec{u}_3,\ \vec{u}_8), & (\vec{u}_5,\ \vec{u}_8) \longleftrightarrow (\vec{u}_6,\ \vec{u}_7), \\
(\vec{u}_1,\ \vec{u}_{10}) \longleftrightarrow (\vec{u}_3,\ \vec{u}_9), & (\vec{u}_2,\ \vec{u}_{10}) \longleftrightarrow (\vec{u}_4,\ \vec{u}_9), \\
(\vec{u}_5,\ \vec{u}_{10}) \longleftrightarrow (\vec{u}_7,\ \vec{u}_9), & (\vec{u}_6,\ \vec{u}_{10}) \longleftrightarrow (\vec{u}_8,\ \vec{u}_9).
\end{array} \tag{5}$$

Let $u = \dfrac{U}{c}$, $v = \dfrac{V}{c}$, $w = \dfrac{W}{c}$ and $e = \dfrac{E}{c^2}$. The microscopic densities of $C1$ in the Maxwellian state associated to the macroscopic variables 1, $\overrightarrow{U} = (U, V, W)$ and E are :

$$\begin{aligned}
N_1 &= \frac{1}{16}\frac{n(1+v)(2e-2u-1)(2e+2w-1)}{2e-1}, & N_6 &= \frac{1}{16}\frac{n(1+v)(2e+2u-1)(2e-2w-1)}{2e-1},\\
N_2 &= \frac{1}{16}\frac{n(1+v)(2e+2u-1)(2e+2w-1)}{2e-1}, & N_7 &= \frac{1}{16}\frac{n(1-v)(2e-2u-1)(2e-2w-1)}{2e-1},\\
N_3 &= \frac{1}{16}\frac{n(1-v)(2e-2u-1)(2e+2w-1)}{2e-1}, & N_8 &= \frac{1}{16}\frac{n(1-v)(2e+2u-1)(2e-2w-1)}{2e-1},\\
N_4 &= \frac{1}{16}\frac{n(1-v)(2e+2u-1)(2e+2w-1)}{2e-1}, & N_9 &= \frac{1}{4}n(1+v)(3-2e),\\
N_5 &= \frac{1}{16}\frac{n(1+v)(2e-2u-1)(2e-2w-1)}{2e-1}, & N_{10} &= \frac{1}{4}n(1-v)(3-2e).
\end{aligned} \tag{6}$$

3 Statement of the problem

We choose the origin O of the orthonormal reference $\Re = (O, \vec{x'}, \vec{y'}, \vec{z'})$ of the physical space so that the plates are located in the planes $y' = -\dfrac{h}{2}$ and $y' = \dfrac{h}{2}$, $h > 0$ (Fig. 1a).

We assume, as is often the case in the treatment of the plane Couette flow in gas dynamics [1, 3, 4, 10] that the flow is one dimensional and depends only upon the spatial variable y' (the normal coordinate in relation to the plate) and the time t'. Due to the symmetry of the model and that of the physical problem, we assume that $N_1 = N_5$, $N_2 = N_6$, $N_3 = N_7$ and $N_4 = N_8$. The independent densities are reduced to N_1, N_2, N_3, N_5, N_9 and N_{10}. The system of kinetic equations describing the evolution of the microscopic densities $N_i(t', y')$, $i \in \{1,\ 2,\ 3,\ 4,\ 9,\ 10\}$, is :

$$\left\{\begin{array}{lcl}
\dfrac{\partial N_1}{\partial t'} + c\dfrac{\partial N_1}{\partial y'} &=& cs\left(\sqrt{2}+\sqrt{3}\right)(N_2N_3 - N_1N_4) + \dfrac{cs\sqrt{6}}{2}(N_3N_9 - N_1N_{10})\\
\dfrac{\partial N_2}{\partial t'} + c\dfrac{\partial N_2}{\partial y'} &=& cs\left(\sqrt{2}+\sqrt{3}\right)(N_1N_4 - N_2N_3) + \dfrac{cs\sqrt{6}}{2}(N_4N_9 - N_2N_{10})\\
\dfrac{\partial N_3}{\partial t'} - c\dfrac{\partial N_3}{\partial y'} &=& cs\left(\sqrt{2}+\sqrt{3}\right)(N_1N_4 - N_2N_3) + \dfrac{cs\sqrt{6}}{2}(N_1N_{10} - N_3N_9)\\
\dfrac{\partial N_4}{\partial t'} - c\dfrac{\partial N_4}{\partial y'} &=& cs\left(\sqrt{2}+\sqrt{3}\right)(N_2N_3 - N_1N_4) + \dfrac{cs\sqrt{6}}{2}(N_2N_{10} - N_4N_9)\\
\dfrac{\partial N_9}{\partial t'} + c\dfrac{\partial N_9}{\partial y'} &=& cs\sqrt{6}\left[(N_1+N_2)N_{10} - (N_3+N_4)N_9\right]\\
\dfrac{\partial N_{10}}{\partial t'} - c\dfrac{\partial N_{10}}{\partial y'} &=& cs\sqrt{6}\left[(N_3+N_4)N_9 - (N_1+N_2)N_{10}\right]
\end{array}\right. \tag{7}$$

Initial and boundary conditions must be prescribed to complete the system (7). The boundary conditions that we consider on the plates are those of the diffuse reflection which allow both exchange of heat and momentum. The forces exerted by the gas on a wall with which it comes into contact and their energy transfers result from the interaction between the gas molecules and those of the wall. When we assume that the wall is impermeable, that is to say it does not emit or absorb particles, it interacts with the gas only through collisions. If such a wall has the velocity $\vec{u}_w$ and the temperature T_w, we define the discrete gas in Maxwellian equilibrium with it at the point M and the time t' as the fictitious gas whose microscopic densities are the strictly positive Maxwellian densities $N_{iw}(M, t')$ associated with the macroscopic variables 1, $\vec{u}_w$ and E_w [3], the total energy E_w is given by the relation $3k_BT_w/2m + \vec{u}_w^2/2 = E_w$.

The velocity distribution and the geometry of the flow are shown in "Figure 1b". The plates are moving in the direction of the x' axis with the velocities $\vec{u}_w^{\pm} = (U_w^{\pm}, 0, 0)$ and with the temperatures $T_w^{\pm}$ in the planes $y = \pm\dfrac{h}{2}$. Denoting by $N_{iw}^{\pm}$, $i = 1, 2, 3, 4, 9, 10$ the microscopic densities of the discrete gas in Maxwellian equilibrium with the plates and by $\Lambda^{\pm}(t')$ the accommodation coefficients (that is the level of acquisition of the macroscopic variables of the plates by the discrete gas at the point $M(t', \pm h/2)$ at the time t'), we have the following relations [9, 10] :

$$\left\{ \begin{array}{lcl} N_i\left(t', -\dfrac{h}{2}\right) & = & \Lambda^-(t')N_{iw}^-, \quad i \in \{1,\ 2,\ 9\} \\ N_i\left(t', +\dfrac{h}{2}\right) & = & \Lambda^+(t')N_{iw}^+, \quad i \in \{3,\ 4,\ 10\} \end{array} \right. \tag{8}$$

The coefficients $\Lambda^-(t')$ and $\Lambda^+(t')$ are determined after the resolution of the initial boundary value problem. The impermeability of the plates means that the normal velocity at the plates vanishes. Therefore :

$$\overrightarrow{U}^- \cdot \vec{n}^- \ = \ 0 \qquad \text{and} \qquad \overrightarrow{U}^+ \cdot \vec{n}^+ \ = \ 0 \tag{9}$$

where $\vec{n}^-$ and $\vec{n}^+$ denote the inward-pointing (i.e into the gas) unit vectors normal to the plates and $\overrightarrow{U}^-$ and $\overrightarrow{U}^+$ the velocities of the discrete gas at $M'\left(t', -\dfrac{h}{2}\right)$ and $M'\left(t', \dfrac{h}{2}\right)$. We can write equations (9) in the form :

$$\left\{ \begin{array}{l} 2\left[cN_1\left(t', -\frac{h}{2}\right) + cN_2\left(t', -\frac{h}{2}\right) - cN_3\left(t', -\frac{h}{2}\right) - cN_4\left(t', -\frac{h}{2}\right)\right] + cN_9\left(t', -\frac{h}{2}\right) - cN_{10}\left(t', -\frac{h}{2}\right) = 0 \\ 2\left[cN_1\left(t', +\frac{h}{2}\right) + cN_2\left(t', +\frac{h}{2}\right) - cN_3\left(t', +\frac{h}{2}\right) - cN_4\left(t', +\frac{h}{2}\right)\right] + cN_9\left(t', +\frac{h}{2}\right) - cN_{10}\left(t', +\frac{h}{2}\right) = 0 \end{array} \right. \tag{10}$$

We assume in addition that the gas is in Maxwellian equilibrium associated to the macroscopic variables N_0, $\overrightarrow{U}_0 = (U_0, V_0, 0)$ and E_0 at the start so the initial conditions are :

$$N_i(0, y') \ = \ N_i^0(y'), \quad \{1,\ 2,\ 3,\ 4,\ 9,\ 10\}. \tag{11}$$

The initial and boundary values problem can be stated as the system of the kinetic equations (7) with the following initial and boundary conditions :

$$\left\{ \begin{array}{l} N_i(0, y') = N_i^0(y'), \quad \{1,\ 2,\ 3,\ 4,\ 9,\ 10\} \\ N_i\left(t', -\frac{h}{2}\right) = \Lambda^-(t')N_{iw}^-, \quad i \in \{1,\ 2,\ 9\} \\ N_i\left(t', +\frac{h}{2}\right) = \Lambda^+(t')N_{iw}^+, \quad i \in \{3,\ 4,\ 10\} \\ 2\left[cN_1\left(t', -\frac{h}{2}\right) + cN_2\left(t', -\frac{h}{2}\right) - cN_3\left(t', -\frac{h}{2}\right) - cN_4\left(t', -\frac{h}{2}\right)\right] + cN_9\left(t', -\frac{h}{2}\right) - cN_{10}\left(t', -\frac{h}{2}\right) = 0 \\ 2\left[cN_1\left(t', +\frac{h}{2}\right) + cN_2\left(t', +\frac{h}{2}\right) - cN_3\left(t', +\frac{h}{2}\right) - cN_4\left(t', +\frac{h}{2}\right)\right] + cN_9\left(t', +\frac{h}{2}\right) - cN_{10}\left(t', +\frac{h}{2}\right) = 0 \end{array} \right. \tag{12}$$

The Maxwellian densities N_{i0} depend on N_0, U_0, V_0 and T_0. Similarly the Maxwellian densities N_{iw} depend on T_w and $\vec{u}_w$. The quantities involved in the physical problem are N_0, U_0, V_0, T_w and $\vec{u}_w$, to which must be added h. Notice that the constant c is a parameter introduced in conjunction with the discrete model. We can choose for c the quantity $\sqrt{3k_BT_w/2m}$, thus c is of the order of magnitude of the speed of sound in the gas.

4 Discretization of the problem

The problem (7) - (11) is put in dimensionless form. Taking the quantities h, N_c, t_c and c as reference values we introduce the dimensionless variables and parameters :

$$\begin{cases} n = \dfrac{N}{N_c}, \quad n_0 = \dfrac{N_0}{N_c}, \quad n_w = \dfrac{N_w}{N_c}, \quad u_w^{\pm} = \dfrac{U_w^{\pm}}{c}, \quad v_w^{\pm} = \dfrac{V_w^{\pm}}{c}, \quad \lambda^{\pm} = \dfrac{\Lambda^{\pm}}{N_c} \\ n_i = \dfrac{N_i}{N_c}, \quad u_i = \dfrac{U_i}{c}, \quad v_i = \dfrac{V_i}{c}, \quad n_{iw}^{\pm} = \dfrac{N_{iw}^{\pm}}{N_c}, \quad i \in \{1,\ 2,\ 3,\ 4,\ 9,\ 10\} \\ u = \dfrac{U}{c}, \quad v = \dfrac{V}{c}, \quad u_0 = \dfrac{U_0}{c}, \quad v_0 = \dfrac{V_0}{c}, \quad y = \dfrac{y'}{h}, \quad t = \dfrac{t'}{t_c} \\ \mathsf{Kn} = (sN_c h)^{-1}, \quad \mathsf{St} = \dfrac{h}{ct_c}, \quad \theta = \dfrac{3k_B T}{2mc^2} = \dfrac{T}{T_w}, \quad \theta_0 = \dfrac{3k_B T_0}{2mc^2} = \dfrac{T_0}{T_w}, \quad \theta_w = \dfrac{3k_B T_w}{2mc^2} = 1 \end{cases} \tag{13}$$

We obtain two dimensionless numbers Kn and St which are respectively the Knudsen number and the Strouhal number. By varying Kn, one passes from continuous gas flows to rarefied gas flows [11]. Clearly $t_0 = \dfrac{h}{c}$ is the characteristic time of propagation of perturbations in the gas i.e the time taken by a perturbation caused by the motion of the moving plate to reach the plate at rest. If we take, as we shall do, t_c to be the characteristic time of unsteadiness, St which is the ratio of the two times controls the transition of the unsteady flow to the steady state and allows to better observe the transition phase.

The problem to solve in the domain $(t,\ y) \in [0, \mathsf{T}] \times \left[-\dfrac{1}{2}, +\dfrac{1}{2}\right]$, $\mathsf{T} > 0$, is :

$$\begin{cases} \mathsf{St}\dfrac{\partial}{\partial t}(n_1) + \dfrac{\partial}{\partial y}(n_1) = \dfrac{(\sqrt{2}+\sqrt{3})}{\mathsf{Kn}}(n_2 n_3 - n_1 n_4) + \dfrac{\sqrt{6}}{2\mathsf{Kn}}(n_3 n_9 - n_1 n_{10}) = Q_1(n) \\ \mathsf{St}\dfrac{\partial}{\partial t}(n_2) + \dfrac{\partial}{\partial y}(n_2) = \dfrac{(\sqrt{2}+\sqrt{3})}{\mathsf{Kn}}(n_1 n_4 - n_2 n_3) + \dfrac{\sqrt{6}}{2\mathsf{Kn}}(n_4 n_9 - n_2 n_{10}) = Q_2(n) \\ \mathsf{St}\dfrac{\partial}{\partial t}(n_3) - \dfrac{\partial}{\partial y}(n_3) = \dfrac{(\sqrt{2}+\sqrt{3})}{\mathsf{Kn}}(n_1 n_4 - n_2 n_3) + \dfrac{\sqrt{6}}{2\mathsf{Kn}}(n_1 n_{10} - n_3 n_9) = Q_3(n) \\ \mathsf{St}\dfrac{\partial}{\partial t}(n_4) - \dfrac{\partial}{\partial y}(n_4) = \dfrac{(\sqrt{2}+\sqrt{3})}{\mathsf{Kn}}(n_2 n_3 - n_1 n_4) + \dfrac{\sqrt{6}}{2\mathsf{Kn}}(n_2 n_{10} - n_4 n_9) = Q_4(n) \\ \mathsf{St}\dfrac{\partial}{\partial t}(n_9) + \dfrac{\partial}{\partial y}(n_9) = \dfrac{\sqrt{6}}{\mathsf{Kn}}[(n_1 + n_2)n_{10} - (n_3 + n_4)n_9] = Q_9(n) \\ \mathsf{St}\dfrac{\partial}{\partial t}(n_{10}) - \dfrac{\partial}{\partial y}(n_{10}) = \dfrac{\sqrt{6}}{\mathsf{Kn}}[(n_3 + n_4)n_9 - (n_1 + n_2)n_{10}] = Q_{10}(n) \\ n_i(0, y) = n_i^0(y), \quad i \in \{1,\ 2,\ 3,\ 4,\ 9,\ 10\} \\ n_i\left(t, -\dfrac{1}{2}\right) = n_{iw}^{-}\lambda^{-}(t), \quad i \in \{1,\ 2,\ 9\} \\ n_i\left(t, +\dfrac{1}{2}\right) = n_{iw}^{+}\lambda^{+}(t), \quad i \in \{3,\ 4,\ 10\} \\ 2\left[n_1\left(t, -\dfrac{1}{2}\right) + n_2\left(t, -\dfrac{1}{2}\right) - n_3\left(t, -\dfrac{1}{2}\right) - n_4\left(t, -\dfrac{1}{2}\right)\right] + n_9\left(t, -\dfrac{1}{2}\right) - n_{10}\left(t, -\dfrac{1}{2}\right) = 0 \\ 2\left[n_1\left(t, +\dfrac{1}{2}\right) + n_2\left(t, +\dfrac{1}{2}\right) - n_3\left(t, +\dfrac{1}{2}\right) - n_4\left(t, +\dfrac{1}{2}\right)\right] + n_9\left(t, +\dfrac{1}{2}\right) - n_{10}\left(t, +\dfrac{1}{2}\right) = 0 \end{cases} \tag{14}$$

where $n = (n_1,\ n_2,\ n_3,\ n_4,\ n_9,\ n_{10})$.

The microscopic densities $n_i(t, y)$, $i \in \{1,\ 2,\ 3,\ 4,\ 9,\ 10\}$, solutions of the system (14) depend on the quantities n_0, u_0, v_0, θ_0, θ_w, n_w and the two dimensionless parameters St and Kn. The problem (14) is solved by digital computation using the fractional steps method [12, 13]. The time step is Δt and the discretised values of the densities are defined by the following scheme :

$$n_i^0 = n_{i0}, \quad i = 1,\ 2,\ 3,\ 4,\ 9,\ 10 \tag{15}$$

$$\begin{cases} \mathsf{St}\dfrac{n_i^{m+\frac{1}{2}} - n_i^m}{\Delta t} = Q_i\left(n^{m+\frac{1}{2}}\right), \quad i = 1,\ 2,\ 3,\ 4,\ 9,\ 10 & (16.1) \\ \mathsf{St}\dfrac{n_i^{m+1} - n_i^{m+\frac{1}{2}}}{\Delta t} + v_i\dfrac{\partial}{\partial y}\left(n_i^{m+1}\right) = 0, \quad i = 1,\ 2,\ 3,\ 4,\ 9,\ 10 & (16.2) \end{cases} \tag{16}$$

$$\begin{cases} n_1^{m+1}n_{2w} - n_2^{m+1}n_{1w} & = 0, \quad y = -\dfrac{1}{2} \\ n_1^{m+1}n_{9w} - n_9^{m+1}n_{1w} & = 0, \quad y = -\dfrac{1}{2} \\ 2\left(v_1 n_1^{m+1} + v_2 n_2^{m+1} + v_3 n_3^{m+1} + v_4 n_4^{m+1}\right) + v_9 n_9^{m+1} + v_{10} n_{10}^{m+1} & = 0, \quad y = -\dfrac{1}{2} \\ n_3^{m+1}n_{4w} - n_4^{m+1}n_{3w} & = 0, \quad y = \dfrac{1}{2} \\ n_3^{m+1}n_{10w} - n_{10}^{m+1}n_{3w} & = 0, \quad y = \dfrac{1}{2} \\ 2\left(v_1 n_1^{m+1} + v_2 n_2^{m+1} + v_3 n_3^{m+1} + v_4 n_4^{m+1}\right) + v_9 n_9^{m+1} + v_{10} n_{10}^{m+1} & = 0, \quad y = \dfrac{1}{2} \end{cases} \tag{17}$$

where n_i^m is the density n_i at time $t = m\Delta$ $(m = 0, 1, 2, \cdots)$, $n_i^{m+\frac{1}{2}}$ the density in middle time and $v_1 = v_2 = -v_3 = -v_4 = v_9 = -v_{10} = 1$. The quantities n_i^m and $n_i^{m+\frac{1}{2}}$ depend only on y.
We make a regular grid of the domain $\left[-\frac{1}{2}, \frac{1}{2}\right]$ with the steps $\Delta y = \dfrac{1}{\mathsf{J}-1}$ where $\mathsf{J} \in \mathbb{N} - \{0,\ 1\}$. Let $n_{i,j}^{m+1}$ be the value of n_i^{m+1} at the point $y \in \left[-\frac{1}{2}, \frac{1}{2}\right]$. We use a finite-difference method to integrate the system (16.2) :

$$n_{i,j+1}^{m+1} - a_i n_{i,j}^{m+1} \;=\; b_i\left(n_{i,j}^{m+\frac{1}{2}} + n_{i,j+1}^{m+\frac{1}{2}}\right), \quad i = 1,\ 2,\ 3,\ 4,\ 9,\ 10 \tag{18}$$

with

$$a_i = \frac{1 - \frac{\mathsf{St}\Delta y}{2v_i\Delta t}}{1 + \frac{\mathsf{St}\Delta y}{2v_i\Delta t}} \quad \text{and} \quad b_i = \frac{\frac{\mathsf{St}\Delta y}{2v_i\Delta t}}{1 + \frac{\mathsf{St}\Delta y}{2v_i\Delta t}}. \tag{19}$$

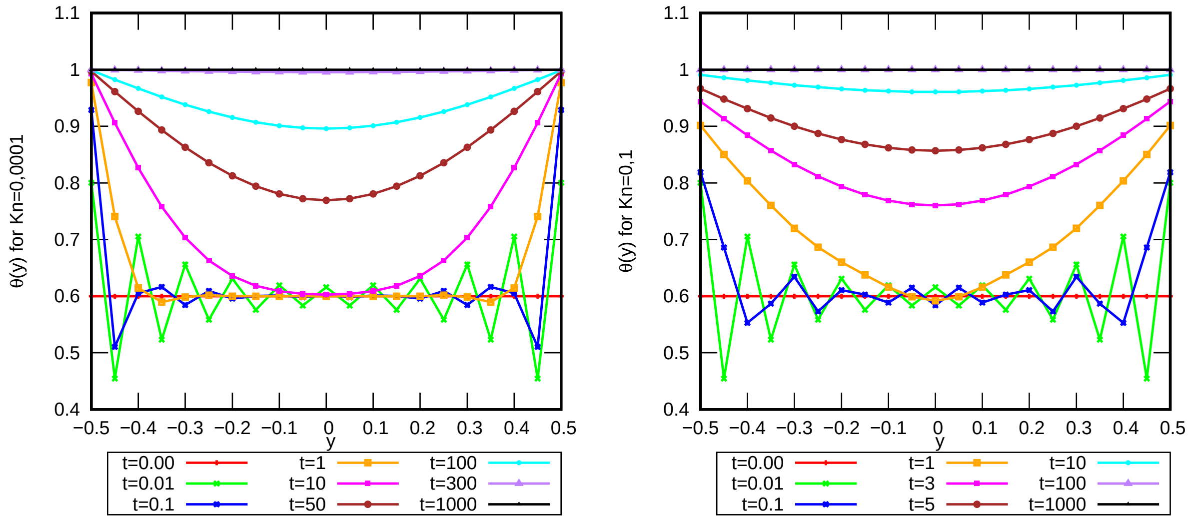
For the computation, we take $\mathsf{J} = 21$ and the time step is $\Delta t = 0,01$ $(\mathsf{T} = 1000)$. The convergence criterion used in the computation is based on the relative error of the macroscopic variables : $\max\left|\dfrac{\phi^{m+1} - \phi^m}{\phi^m}\right| \leqslant 10^{-6}$, where $\phi = u,\ \theta$.

5 Numerical results

In the numerical resolution, we set $n_0 = n_w = 1$, where n_0 is the total density of the gas at the initial time and $n_w = 1$ is the total density of the discrete gas in the Maxwellian equilibrium with the plate. We take the number of Strouhal $\mathsf{St} = 2,5$. We assume in the sequel that the two plates have the same dimensionless temperature $\theta_w^- = \theta_w^+ = 1$.

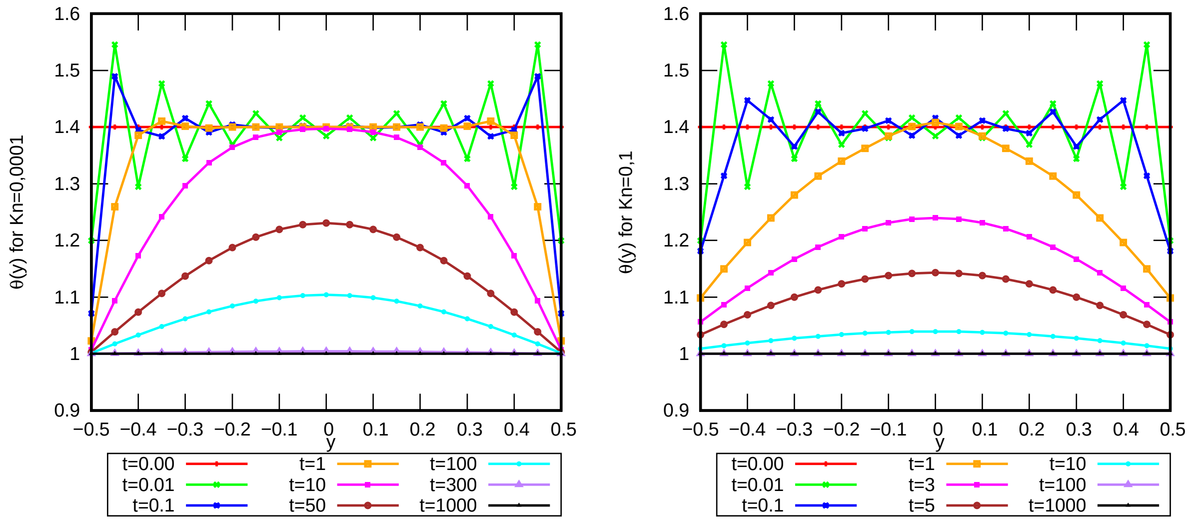
5.1 The plates are at rest

When the plates are at rest and the gas is initially at rest with a temperature different from that of the plates, there is just a heat transfer process : the total density is constant and the macroscopic velocity remains zero throughout the gas flow. Only the temperature of the gas varies so as to establish the thermal equilibrium with the plates as indicated in the figures below : when the gas is initially at a lower temperature than that of the plates ("Figure 2a" and "Figure 2b"), and when the initial temperature of the gas is higher than that of the plates ("Figure 3a" and "Figure 3b") for $\mathsf{Kn} = 0,0001$ and $\mathsf{Kn} = 0,1$. In all cases the kinetic temperature of the discrete gas in the flow is symmetrical with respect to the $y = 0$ axis, and when the flow becomes permanent there is no temperature jump on the plates.



(a) $\mathsf{Kn} = 0,0001$ (b) $\mathsf{Kn} = 0,1$

FIGURE 2 – Kinetic temperature when the plates are immobile ($\theta_0 < \theta_w^{\pm}$)



(a) $\mathsf{Kn} = 0,0001$ (b) $\mathsf{Kn} = 0,1$

FIGURE 3 – Kinetic temperature when the plates are immobile ($\theta_0 < \theta_w^{\pm}$)

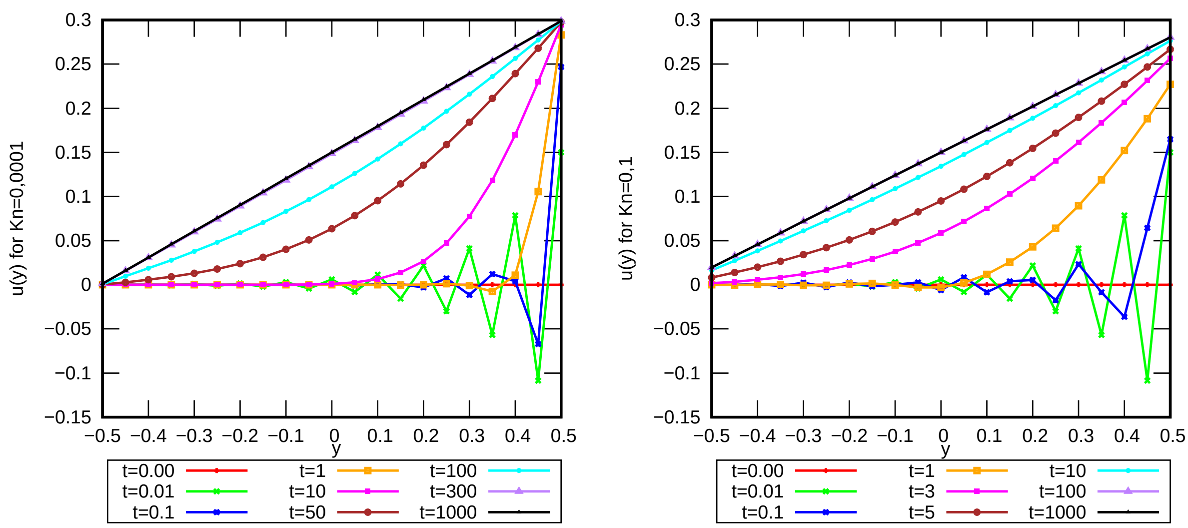
5.2 One of the plate is moving

We study here the case where the gas is initially at rest with a temperature lower than that of the plates. The initial values of the macroscopic variables of the gas are $u_0 = 0$, $v_0 = 0$ and $\theta_0 = 0,6$.

The macroscopic variables on the plates are : in $y = -\frac{1}{2}$, $u_w^- = 0$, $v_w^- = 0$, $\theta_w^- = 1$ and in $y = \frac{1}{2}$, $u_w^+ = 0,3$, $v_w^+ = 0$, $\theta_w^+ = 1$.

5.2.1 Longitudinal velocity

The motion of the plate induces fluctuations of the longitudinal velocity at the beginning of the transition phase. The intensity of these fluctuations decreases from the vicinity of the moving plate towards the fixed plate ("Figure 4a" and Figure 4b"). In the transition phase, the longitudinal velocity increases as one approaches the moving plate. The longitudinal velocity which is initially zero gradually increases to a constant value reached at the steady state. This value depends on the degree of rarefaction of the gas (Fig. 5). In the continuous flow regime, the non-slip condition holds : the longitudinal velocity of the gas in the vicinity of each plate equals the velocity of the plate ("Figure 4a", $t = 1000$). In the transition regime, there is velocity slip at the wall : in the vicinity of the fixed plate the velocity of the gas is strictly greater than zero and in the vicinity of the moving plate this velocity is strictly lower than the velocity of this plate ("Figure 4b", $t = 1000$). The longitudinal velocity of the gas at $y = 0$ is equal to the average of the plate velocities and its profile is linear, when the flow becomes permanent, whatever the flow regime (Fig. 5). The results obtained with the $C1$ model show the dependence of the velocity slip upon the Knudsen number Kn in the steady state. At the moving plate, the velocity slip decreases and tends to 0 with Kn. However, its variation rate is not uniform. The variation is fast for small values of Kn and low for large values of Kn (Fig. 6). The velocity slip tends to the half of the difference of the velocities of the plates when Kn tends to infinity. This confirms the results obtained with the Broadwell four velocity model [14].



(a) Kn $= 0,0001$ (b) Kn $= 0,1$

FIGURE 4 – Longitudinal velocity as a function of y

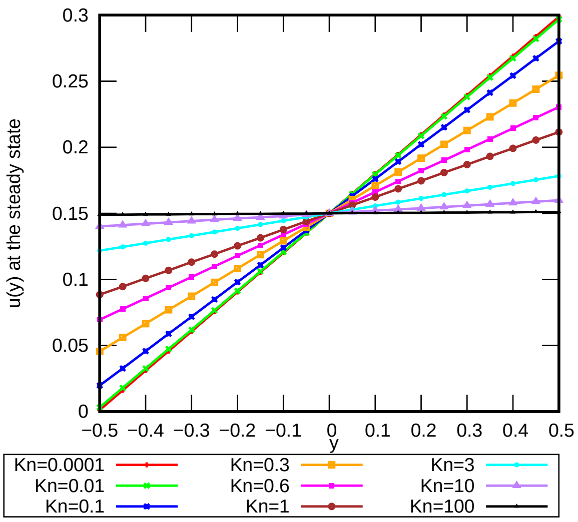


FIGURE 5 – Longitudinal velocity at steady state as a function of y

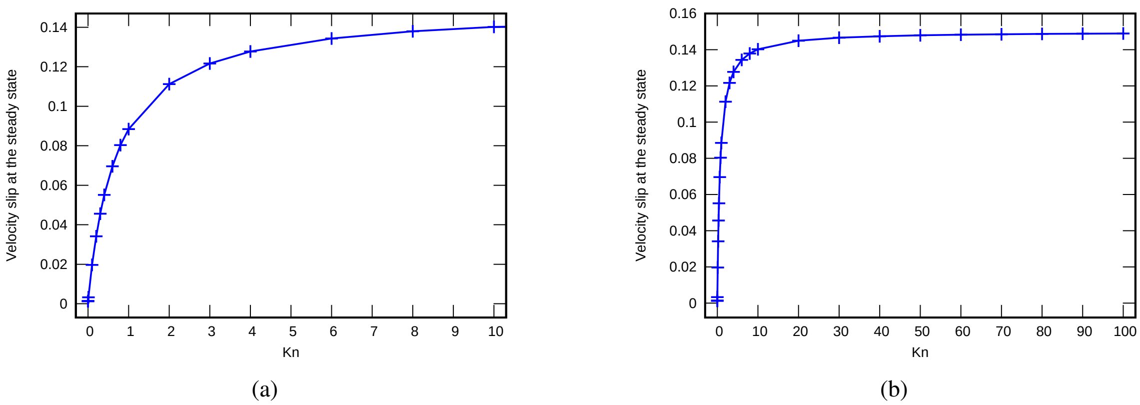


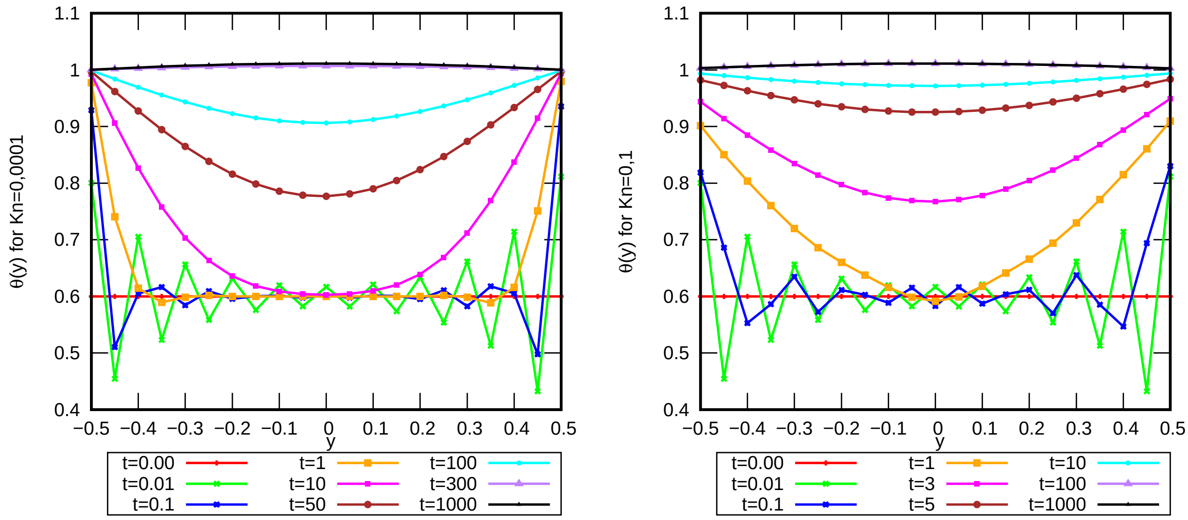
FIGURE 6 – Velocity slip at steady state as a function of Kn

5.2.2 Kinetic temperature

The temperature of the plates is higher than that of the gas at the initial time. There is therefore a warming of the gas by the plates. At the beginning of the transition phase, we observe the fluctuations of the kinetic temperature due to the energy used to ensure the movement of the gas (Fig. 7). These fluctuations disappear over time in the transition phase.

At the steady state the profile of the kinetic temperature of the discrete gas depends on the degree of rarefaction of the gas (Fig. 8). Hence the profile is parabolic in the transitional and slipping regimes, and linear in the rarefied and highly rarefied regimes. These profiles of the kinetic temperature are symmetrical with respect to the $y = 0$ axis.

The temperature jump depends on Kn and on the velocity of the moving plate (Fig. 9). It decreases and tends to 0 with Kn but with a non-uniform variation rate. The variation is fast for small values of Kn and low for large values of Kn. When Kn tends towards $+\infty$ temperature jump tends towards a constant.



(a) Kn $= 0,0001$ (b) Kn $= 0,1$

FIGURE 7 – Kinetic temperature as a function of y

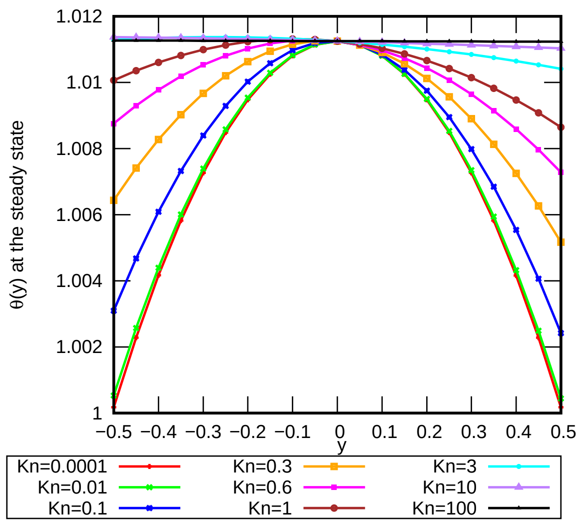
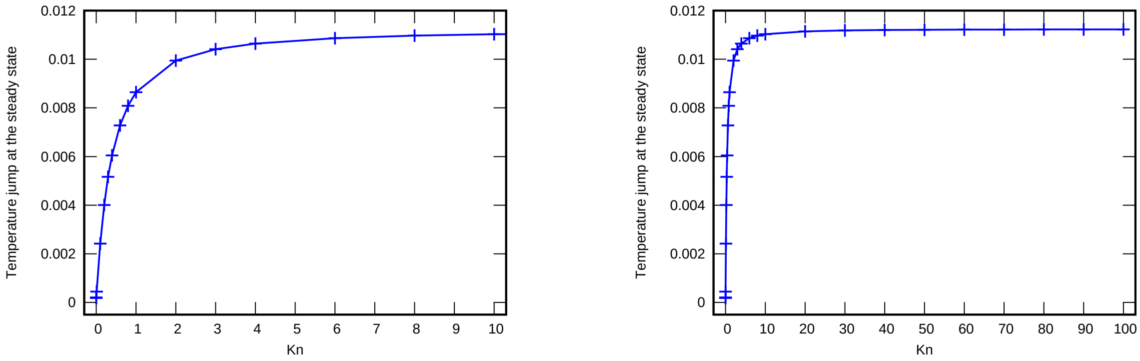


FIGURE 8 – Kinetic temperature at steady state as a function of y

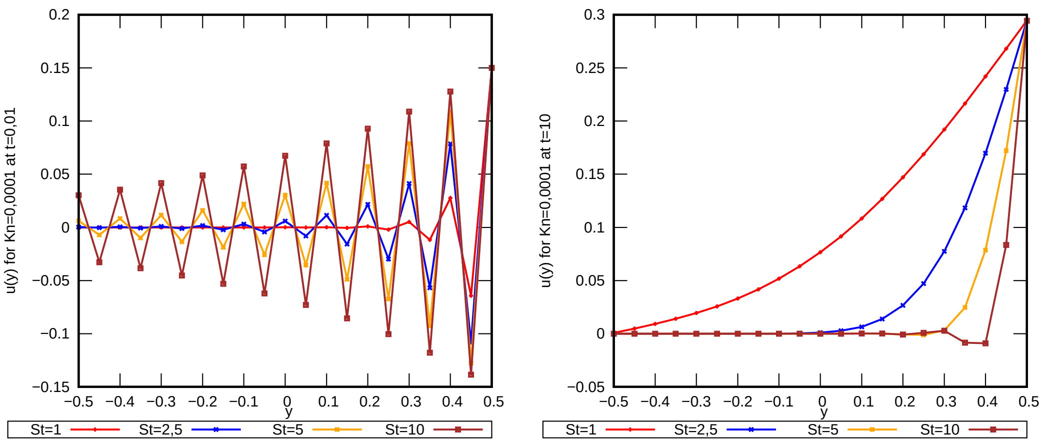


(a) (b)

FIGURE 9 – Temperature jump at teady state as a function of Kn

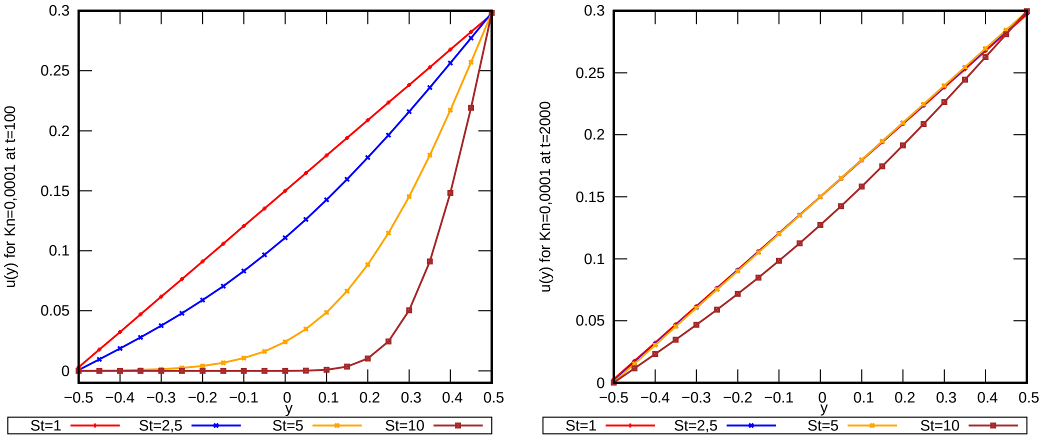
5.3 Effect of the Strouhal number St

We analyse the influence of the Strouhal number St on the longitudinal velocity and the kinetic temperature in the case of a continuous flow (Kn $= 0,0001$). The macroscopic variables of moving and fixed plate are respectively $u_w^+ = 0,3$, $v_w^+ = 0$ and $u_w^- = 0$, $v_w^- = 0$; the discrete gas initially having a kinetic temperature ($\theta_0 = 0,6$) lower than that of the plates ($\theta_w^- = \theta_w^+ = 1$).
We take $t_0 \geqslant t_c$ so St $\geqslant 1$. When $t_0 = t_c$ i.e St $= 1$ the steady state is attained faster than for $t_0 > t_c$ that is St > 1. Thus to better observe the transition phase one can increases the Strouhal number St.
At the beginning of the transition phase amplitudes of fluctuations increase with St ("Figure 10a" and "Figure 12a"). In the transition phase, longitudinal velocity and kinetic temperature increase rapidly as St decreases ("Figure 10b" - "Figure 11a and "Figure 12b" - "Figure 13a). The steady state is reached faster when St decreases. For example, with the kinetic temperature at $t = 2000$, we are already in the permanent state for St $= 1$, St $= 2,5$ and St $= 5$ (Fig. 13b), but not yet for and St $= 10$ for which the steady state is reached after $t = 2000$. It's the same for the longitudinal velocity (Fig. 11b).



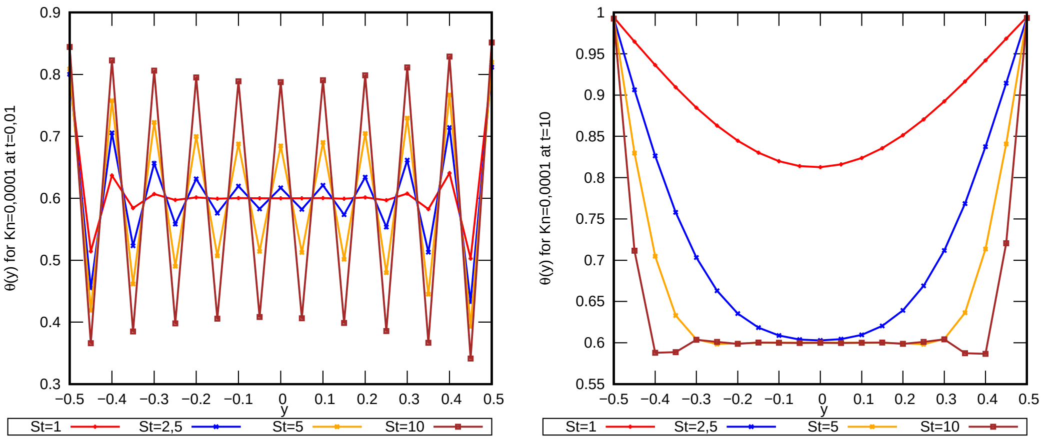
(a) $t = 0,01$ (b) $t = 10$

FIGURE 10 – Longitudinal velocity as a function of y (Kn $= 0,0001$)



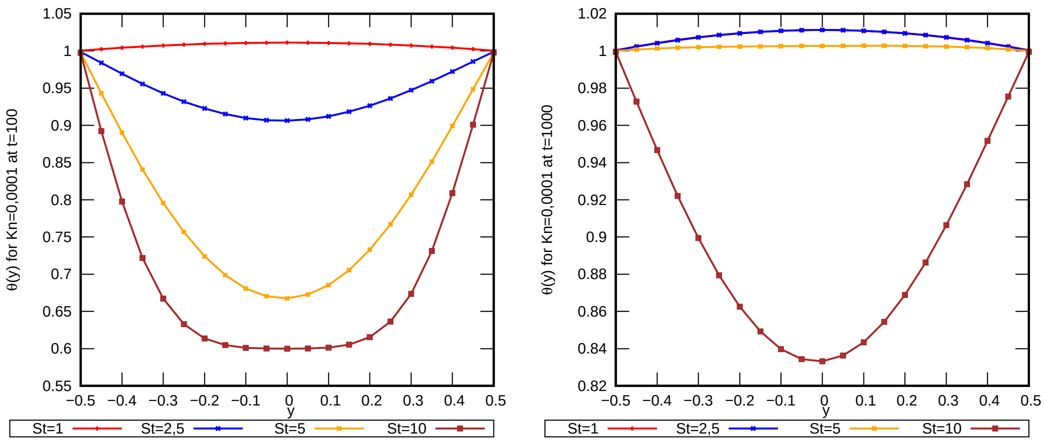
(a) $t = 100$ (b) $t = 2000$

FIGURE 11 – Longitudinal velocity as a function of y ($\mathsf{Kn} = 0,0001$)



(a) $t = 0,01$ (b) $t = 10$

FIGURE 12 – Kinetic temperature as a function of y ($\mathsf{Kn} = 0,0001$)



(a) $t = 100$ (b) $t = 2000$

FIGURE 13 – Kinetic temperature as a function of y ($\mathsf{Kn} = 0, 0001$)

6 Conclusion

We have solved numerically using the fractional step method the problem of the unsteady flow of a discrete gas between two parallel infinite moving plates with the same temperature. Since the transverse velocity of the plates is zero, the transverse velocity of the flow after a few small fluctuations at the beginning of the transition phase vanishes in all the flow regimes. The results reported here are related to the cases where the initial temperature of the discrete gas in the flow is lower than the temperature of the plates and the plates are both at rest or one moving and the other at rest.

When the plates are at rest we have a pure heat tranfer between the discrete gas and the plates. In the transition phase the temperature of the flow is always lower than that of the plates and has a parabolic profile with a minimum in $y = 0$. At th steady state, there is no velocity slip or temperature jump whatever the flow regime, and the temperature of the flow is constant and equal to temperature of the plates.

When one of the plate is moving the longitudinal velocity has a non monotonous parabolic profile at the transition phase. In the steady state the profile is linear. However velocity slip exists and depends on Kn and the velocities of the plates. It tends to zero when Kn tends to zero. The kinetic temperature of the discrete gas, after fluctuations at the beginning of the transition phase, changes from a non monotonous parabolic profile with a minimum in the flow to a non monotonous profile with maxima in the flow higher than the temperature of the plates. This profile is kept in the steady state. The temperature jump depends on Kn and the velocity of the moving plate and is an increasing function of the velocity of the moving plate for fixed Kn. Despite their simplicity discrete velocity models reflect the complex dynamics of gas flows.

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