ANM continuation with efficient vectorial Padé approximants

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Abstract :

In this work, we investigate numerically efficient vectorial Padé approximants in the Asymptotic Numerical Method (ANM). These efficient vectorial Padé approximants are deduced from a new matrix generalized definition of vectorial Padé representations. A comparison between the proposed vectorial Padé approximants and the classical Padé approximants will be presented. Another hand, we propose a new analytical formula of the end step of vectorial Padé approximants. The ability of the proposed continuation will be demonstrated on several examples of post-buckling shells.

Keywords : Vectorial Padé approximants, Asymptotic Numerical Method, Matrix generalized definition.

1 Introduction

In the previous work of the ANM and in order to increase the validity range of the Taylor series, one often uses vectorial Padé approximants [1]. The most commonly used technique consists to orthonormalize the terms of Taylor series and to replace the polynoms that appear by Padé approximants having the same denominator. Classically, the validity range of Padé approximants is determined numerically by the dichotomy method to minimize the relative error between two consecutive Padé approximants [5].

In this work, we investigate numerically efficient vectorial Padé approximants in the ANM derived from a new matrix generalized definition of vectorial Padé approximants. Note that there is a connection between this type of vectorial Padé approximant and the vectorial Padé introduced in the reference [6].

We will also discuss a new analytical definition of the end of the step of vectorial Padé approximants.

The numerical invistagation of the proposed vectorial Padé approximants and the validity range between the analytical formula and that obtained numerically by dichotomy method is presented on the example of a cylindrical shell buckling. The structure studied is discretized using the Finite Element Method (FEM) in DKT18 element [2].

2 Matrix generalized definition of vectorial Padé representations

In general, most engineering problems can be reduced to solving non-linear problems of the following form :

$$\{R\}(\{U\}, \lambda) = \{0\} \tag{1}$$

where $\{R\}$ is a regular vectorial function with values in $\mathbb{R}^n$, $\{U\}$ denotes the unknown vector of $\mathbb{R}^n$ and λ represents a control parameter. The resolution of (1) by the Asymptotic Numerical Method (ANM) consists to approach $\{U\}$ and λ by a truncated vectorial series at a given order N with respect to a scalar parameter "a" of the validity range $[0, a^s_{max}]$, whose expression is given explicitly with respect to the first and the last terms of the Taylor series [1].

$$\begin{aligned} &\{U\}(a) = \{U^j\} + \textstyle\sum_{i=1}^{N} a^i \{U_i\} \\ &\lambda(a) = \lambda^j + \textstyle\sum_{i=1}^{N} a^i \lambda_i \end{aligned} \qquad a \in [0, a^s_{max}] \tag{2}$$

where $(\{U^j\}, \lambda^j)$ represents the solution of the end of the previous branch j. In order to expand the validity range $[0, a^s_{max}]$, we propose in this work, as in a previous paper [3], to replace the Taylor series with vectorial Padé approximants defined over an interval $[0, a^p_{max}]$ written under the following matrix form :

$$\{U[L, M]\}(a) \quad = \quad (\textstyle\sum_{m=0}^{M} a^m [B_m])^{-1} (\sum_{l=0}^{L} a^l \{A_l\}) \tag{3}$$

where $[B_0] = [I_n]$, L and M are integers such that $N = L + M$ and $L \leq M$, the $[B_m]$ are square matrices of order n and the $\{A_l\}$ are vectors of $\mathbb{R}^n$. As in the scalar case $\{U[L, M]\}(a)$ must have the same development as $\{U\}(a)$ until the order N and must satisfy the following matrix equation :

$$-\{A\}(a) + [B](a)\{U[L, M]\}(a) = 0 \tag{4}$$

where the vector $\{A\}(a)$ and the matrix $[B](a)$ are defined by : $\{A\}(a) = \sum_{l=0}^{L} a^l \{A_l\}$, $[B](a) = \sum_{m=0}^{M} a^m [B_m]$.

The functions $\{A\}(a)$ and $[B](a)$ are deduced from the relationship

$$-\{A\}(a) + [B](a) \sum_{i=1}^{L+M} a^i \{U_i\} = O\left(a^{L+M+1}\right)$$

which are solutions of the following systems, for $L + 1 \leq l \leq L + M$ and $1 \leq j \leq L$ the equalities [3]

$$\textstyle\sum_{m=1}^{min(l,M)} [B_m]\{U_{l-m}\} = -\{U_l\}, \;\; \{A_j\} = \sum_{m=1}^{j} [B_m]\{U_{j-m}\}. \tag{5}$$

Solving the system (5), the new definition of the proposed family of vectorial Padé Approximants can be written as in [3] in the following form :

$$\{U[L, M]\}(a) \quad = \quad \textstyle\sum_{l=0}^{L} a^l \{U_l\} + a^L \sum_{m=1}^{M} \frac{\Delta_{M-m}(a)}{\Delta_M(a)} a^m \{U_{L+m}\} \quad ; \quad a \in [0, a^p_{max}] \tag{6}$$

where the polynôms are given by $\Delta_M(a) = \sum_{i=0}^{M} b_i a^i$, with b_0 equal to 1 and b_i, $1 \leq i \leq M$ are computed by a Gram-Schmidt orthonormalization thechnique of the vectors $\{U\}$ [1], [3].

In the next section, we will discuss numerically the efficiency of this new rational representation in ANM continuation algorithm.

3 Numerical experimentations

We consider a cylindrical shell articulated along the two opposite edges $(1; 2)$ and free on the two others $(3; 4)$. This structure is of length $2L = 504mm$, radius $R = 2540mm$ and angle of half opening $\theta = 0.1rad$, made of a homogeneous, elastic and isotropic material of Young modulus $E = 3102.75MPa$ and Poisson ratio $\nu = 0.3$, subjected to a vertical loading λF applied at its center with $F = 1000N$.

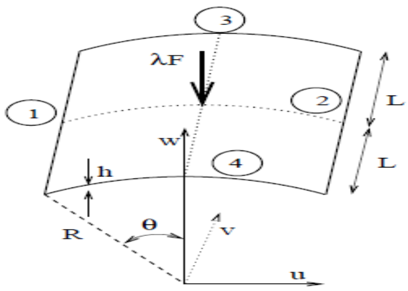


FIGURE 1 – Geometrical characteristics of the studied structure loaded at its center with λF

The studied structure has the thickness $h = 12.7mm$. Due to the symmetry, only a quarter of the structure is discretized in 200 triangular elements of type $DKT18$ i.e. 726 degrees of freedom. In the following, two types of continuations based on vectorial Padé approximants are adopted ; the continuation based on the determination of a^p_{max} by the dichotomy method and that based on the determination of a^p_{max} in an analytical way.

3.1 Continuation with the dichotomy method

Classically, the parameter a^p_{max} is computed numerically by the resolution of the following equation [5] :

$$\frac{||\{U_P^N\}(a) - \{U_P^{N-1}\}(a)||_2}{||\{U_P^N\}(a) - \{U^j\}||_2} - \varepsilon_p = 0 \tag{7}$$

where ε_p is the tolerance parameter, $\{U_P^N\}(a)$ and $\{U_P^{N-1}\}$ two consecutive Padé approximants of order N and $N-1$ respectively.

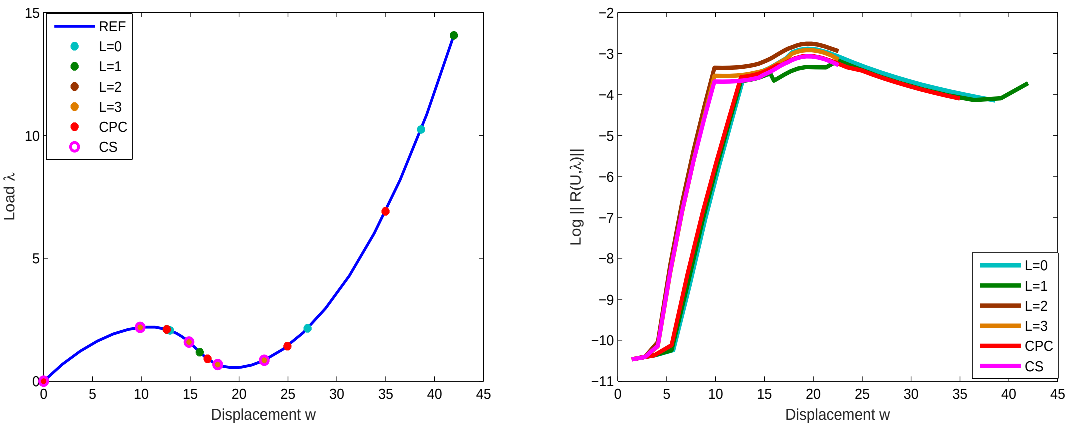
In this section, we compare the numerical results obtained by the continuation with Taylor series noted by CS, the continuation with classical Padé approximants CPC [5] and the continuation with vectorial Padé approximants deduced from (5) for $L = 0, 1, 2, 3$. The parameter a^p_{max} is computed numerically by the dichotomy method. This comparison is made with the same quality of solution for the three types of continuations. For this, we fix the truncation order $N = 15$, the number of ANM step $Nstep = 4$ and we choose the tolerance parameters for each algorithm so as to have the same quality of solution.

In table 1, we represent the attainted values of λ and w at the loaded node during 4 ANM steps. From this table, we note that when we use a continuation CS we reach a deflection $w = 22.57mm$. On the other hand, if we choose a continuation CPC, we reach a deflection $w = 34.99mm$. If we use the continuation for $L = 0$, we reach a deflection $w = 38.61mm$ and for $L = 1$ we reach a deflection $w = 41.97mm$ whereas for $L = 2$ and $L = 3$ we reach deflections respectively $w = 22.18mm$ and $w = 22.05mm$. From these results, we note that the continuation for $L = 0$ and $L = 1$ allows to have a high deflection that the other continuations and for $L = 2$ and $L = 3$, the solution is degraded and it becomes close to the continuation with the series, this is certainly due to the fact that for these cases, the polynomial part is much larger.

$N = 15$	CS		CPC		$L = 0$		$L = 1$		$L = 2$		$L = 3$	
Tolerance parameter	$\varepsilon_s = 5\ 10^{-6}$		$\varepsilon_p = 16\ 10^{-6}$		$\varepsilon_p = 17\ 10^{-6}$		$\varepsilon_p = 9.1\ 10^{-6}$		$\varepsilon_p = 10^{-8}$		$\varepsilon_p = 10^{-8}$	
$ANM\ step = 4$	λ	w	λ	w	λ	w	λ	w	λ	w	λ	w
	0.84	22.57	6.91	34.99	10.24	38.61	14.07	41.97	0.77	22.18	0.75	22.05

Table 1 – ANM continuation, 4 steps : Comparison between the results obtained with Taylor series CS, classical Padé CPC and proposed Padé approximant for $L = 0, 1, 2, 3$, using a given tolerance parameter

In figure 2, we represent the comparison between the considered continuations in plane (w, λ) and $(w, ||R||)$. From this figure, we remark a good agreement between the different continuations, the cases $L = 0$ and $L = 1$ give better results.



(a) Load-deflection curve

(b) Residual-deflection curve

Figure 2 – Load-displacement and residual-deflection curves for $N = 15$ obtained in 4 ANM steps with Taylor series CS, classical Padé CPC and proposed Padé approximant for $L = 0, 1, 2, 3$

3.2 Continuation with analytical formula

We propose in this work an analytical approximation of a^p_{max}. To determine this approximation, we minimize the relative error between two consecutive Padé approximants (7) by its Taylor development with respect to the parameter "a" truncated at order 3 or 4. We then determine all the analytical real roots of the polynomial of degree 3 or 4 and we approach the parameter a^p_{max} by the smallest of these roots.

To establish an analytical formula of the step length a^p_{max}, the formula (7) can be rewritten in the following matricial form

$$E_M(a) - \varepsilon_p = a^{M-1} \frac{||[R_M] \sum_{k=0}^{M-1} a^k \{\varphi_k^M\}||_2}{||[R_M] \sum_{k=0}^{2M-2} a^k \{\eta_k^M\}||_2} - \varepsilon_p = 0 \qquad (8)$$

where the matrix $[R_M] = (\alpha_{ij})_{1 \leq i \leq j \leq M}$ is an upper triangular square matrix of order M obtained by a factorization method QR from the matrix $[U_M]$ whose columns are the terms of the initial vectorial series, with α_{ij} are the orthogonalization coefficients of Gram-Schmidt, the vectors $\{\varphi_k\}$, $0 \leq k \leq M-1$ and $\{\eta_k\}$, $0 \leq k \leq 2M-2$ are functions of the coefficients b_k^M $(1 \leq k \leq M)$ of the polynomial Δ_k^M of the Padé approximants truncated at orders M and $M-1$ [4].

If we develop the equation (8) with respect to the parameter "a" at order 3 or 4 we obtain respectively the both following polynoms [4] :

$$\begin{aligned} P_3(a) &= A_3 a^3 + A_2 a^2 + a - A_1 \\ P_4(a) &= A_4 a^4 + A_3 a^3 + A_2 a^2 + a - A_1 \end{aligned} \tag{9}$$

where A_i, $1 \leq i \leq 4$ are coefficients which depend on b_k, N and ε_p. We denote $\tilde{a}^p_{max}$ and $\hat{a}^p_{max}$ the minimal roots respectively of $P_3(a)$ and $P_4(a)$.

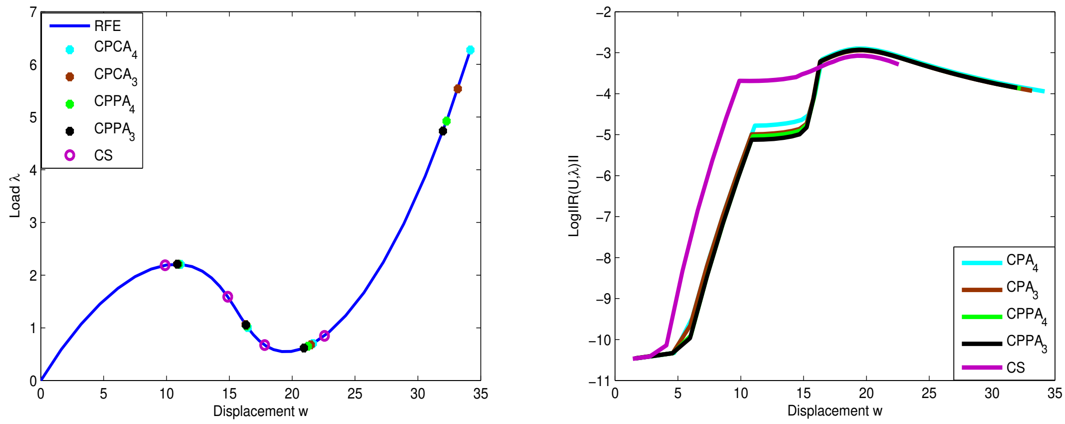
We consider the same numerical example and we compare three algorithms, the continuation with Taylor series (2) CS, the continuation with the classical Padé approximant [1] CPC and the continuation with the proposed Padé approximant for $(L = 1)$ CPP. The validity range of Padé approximants is estimated by $\tilde{a}^p_{max}$ or $\hat{a}^p_{max}$. For this, we fix the truncation order $N = 15$, the number of ANM steps $Nstep = 4$ and we choose the tolerance parameters for each algorithm to have the same quality of solution.

We report in table 2 the attainted values of λ and w at the loaded node during 4 ANM steps. From this table, we note that if we choose a continuation CPC, we reach a deflection $w = 33.17mm$ with $\tilde{a}^p_{max}$ (CPA_3) and $w = 34.18$ with $\hat{a}^p_{max}$ (CPA_4) and if we choose the continuation CPP, we reach a deflection $w = 31.99mm$ with $\tilde{a}^p_{max}$ $(CPPA_3)$ and a deflection $w = 32.28mm$ with $\hat{a}^p_{max}$ $(CPPA_4)$. From these results, we note that the continuations CPC and CPP with $\tilde{a}^p_{max}$ or $\hat{a}^p_{max}$ for this numerical example, results cloose to the method of dichotomie.

$Norder = 15$	CPC				CPP			
	$\tilde{a}^p_{max}$		$\hat{a}^p_{max}$		$\tilde{a}^p_{max}$		$\hat{a}^p_{max}$	
Tolerance parameter	$\varepsilon_p = 10^{-4}$		$\varepsilon_p = 2\ 10^{-4.1}$		$\varepsilon_p = 10^{-4.3}$		$\varepsilon_p = 2\ 10^{-4.5}$	
$Nstep = 4$	λ	w	λ	w	λ	w	λ	w
	5.53	33.17	6.27	34.18	4.73	31.99	4.92	32.28

TABLE 2 – Comparison between the results obtained with classical Padé CPC and proposed Padé approximant CPP for $L = 1$, using the analytical formula of a^p_{max}

We represent in figure 3, the comparison between the considered continuations in plane (w, λ) and $(w, ||R||)$. From this figure, we remark a good agreement between the different continuations.



(a) Load-deflection curve (b) Residual-deflection curve

FIGURE 3 – Load-displacement and residual-deflection curves for $N = 15$ obtained by 4 ANM steps with CS, classical Padé CPC using $\tilde{a}^p_{max}$ $(CPCA_3)$, $\hat{a}^p_{max}$ $(CPCA_4)$ and proposed Padé approximant CPP for $L = 1$ using $\tilde{a}^p_{max}$ $(CPPA_3)$, $\hat{a}^p_{max}$ $(CPPA_4)$

4 Conclusion

In this work, we have shown the efficiency of a class of vector Padé approximants compared to the classical Padé approximant on an example of buckling. Thus, we have proposed a Padé continuation with an analytic formula, which allows us to say that this new class of Padé approximant combined with an analytic continuation gives good results.

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