Une approche optimale de type éléments finis pour écoulements visqueux incompressibles dans des domaines courbes

An optimal finite-element technique for viscous incompressible flow in curved domains

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Résumé :

La théorie mathématique de la méthode des éléments finis pour la simulation d'écoulements visqueux incompressibles est un sujet bien maîtrisé depuis près de quarante ans. Cependant le plus souvent les analyses sous-jacentes supposent d'une part que le domaine de l'écoulement est un polytope, et d'autre part que la solution exacte est régulière. Ces hypothèses étant plutôt contradictoires on s'emploie à démontrer des résultats de convergence optimaux s'appliquant au cas où la vitesse est imposée sur toute la frontière d'un domaine d'écoulement régulier. Plus précisément ces résultats concernent des méthodes d'ordre deux ou plus dans la norme H^1 pour la vitesse et la norme L^2 pour la pression, basées sur des triangles ou des tétraèdres. Le principal ingrédient de cette démarche est la prise en compte des valeurs connues de la vitesse aux noeuds situés sur la frontiére courbe, qu'ils soient des sommets ou pas, à l'instar de la technique isoparamétrique. En revanche ici les fonctions-test sont polynomiales par élément s'annullant sur toute la frontiére du polytope approchant le domaine courbe, constitué des simplexes droits du maillage utilisé. En fait les fonctions de forme sont polynomiales par morceaux également, néanmoins ne vérifiant des conditions aux limites que sur la vraie frontière. De la sorte on peut se passer des éléments courbes sans perte de la qualité d'approximation. Des exemples sur des méthodes de Galerkin classiques telles la méthode de Taylor-Hood et la méthode conforme de Crouzeix-Raviart, servent à illustrer le bien fondé de l'approche préconisée. En outre la convergence optimale en norme L^2 de la vitesse est démontrée. À la connaissance de l'auteur ces derniers résultats sont inédits, dans le cadre de l'approximation de problèmes aux limites posés dans des domaines courbes par éléments finis d'ordre deux ou plus en normes de Sobolev naturelles.

1. Bénéficiaire d'un contrat de recherche avec le Conseil Brésilien de la Recherche Scientifique et Technique - CNPq. Research grant holder of CNPq - the Brazilian Council for Scientific and Technological Research.

Abstract :

The mathematical theory of the finite element method for the simulation of viscous incompressible flow is mastered since over three decades in its basic aspects. However in genral the underlying analyses are carried out under the assumptions that the flow domain is a polytope, and that the exact solution is smooth. Since both hypotheses are rather contradictory, we endeavor to establish optimal convergence results for the case where the velocity is prescribed all over the boundary of a smooth flow domain. More precisely such results apply to triangle- or tetrahedron-based methods of order two or more in the H^1-norm for the velocity and the L^2-norm for the pressure. The main feature of our approach is the enforcement of prescribed velocity values at nodes on the curvilinear boundary, instead of the boundary of an approximating polytope, akin to method's isoparametric version. However in contrast to the latter, in the former the test-functions are piecewise polynomials that vanish everywhere on the boundary of this polytope - that is, the union of the straight-edged simplexes of the mesh in use. Incidentally the shape-functions are also piecewise polynomials that nevertheless fulfill boundary conditions only at nodes located on the true boundary. In this way it is possible to rule out curved elements without order erosion. Examples of classical Galerkin methods such as Taylor-Hood's and the conforming Crouzeix-Raviart mixed method for triangles illustrate the strength of the new approach. Furthermore optimal convergence of the velocity in the L^2-norm is demonstrated. To the best of author's knowledge such results are unprecedented in the framework of the approximation of boundary value problems in curved domains by finite element methods of order two or more in natural Sobolev norms.

Mots clefs : Domaine courbe ; écoulement incompressible ; élément fini ; fluide visqueux ; vitesse imposée.

Keywords : Curved domain ; finite element ; incompressible flow ; prescribed velocity ; viscous fluid.

1 Introduction

In the framework of the finite-element solution of second order boundary value problems posed in curved domains with Dirichlet conditions, it is well known that a considerable order lowering may occur if prescribed boundary values are shifted to nodes that are not mesh vertexes of an approximating polygon or polyhedron formed by the union of straight-edged N-simplexes of a fitted mesh. Over four decades ago some techniques were designed in order to remedy such a loss of accuracy, and possibly attain the same theoretical optimal orders as in the case of a polytopic domain, assuming that the solution is sufficiently smooth. Two examples of such attempts in the framework of two-dimensional problems are the interpolated boundary condition method by Nitsche and Scott (cf. [10] and [18]), and the method introduced by Zlámal in [23] and extended by Ženíšek [21]. Among such techniques the finite element method's isoparametric version is by far the one most widely in use since the sixties (cf. [22]) in order to recover the lost optimality. One of the main reasons why it became so popular is the fact that the isoparametric technique applies to both two- and three-dimensional problems. We recall that this version of the finite element method is based on elements with curved boundary portions, aimed at better

approximating a curved boundary than straight edges or plane faces. In this case the aforementioned shift of prescribed boundary values is avoided, since all nodes to which such values apply remain on the true boundary. The price to pay however is the manipulation of rational functions as both shape- and trial-functions defined upon the curved elements, and the resulting compulsory use of numerical integration. While on the one hand this is far from being an obstacle in most current applications such as linear problems with constant coefficients, numerical integration can be a delicate issue in more complex situations. The technique exploited in this work allows overcoming all such issues, since it is based only on polynomial algebra upon an ordinary (i.e. a straight-edged) N-simplex. Moreover, in contrast to the simple polygonal approach no erosion of the theoretical order of a given interpolation inherent to the method occurs, for methods which are not of the lowest possible order. In short, our technique is aimed at ensuring a theoretical order greater or equal to two in the natural norm, without the use of curved elements and interpolating functions other than piecewise polynomials.
Actually the conception of the finite-element technique for solving boundary value problems with a smooth curvilinear boundary considered in this work is close to the interpolated (Dirichlet) boundary condition method studied in [4]. Though intuitive and known since the seventies, the latter technique has been of limited use so far. Among the reasons for this lies its difficult implementation, the lack of an extension to three-dimensional problems and restrictions on the choice of boundary nodal points to reach optimal convergence rates. In contrast our method is simple to implement in both two- and three-dimensional geometries. Moreover it is particularly handy, whenever a finite element method has normal component or normal derivative degrees of freedom as illustrated in [14] and [17]. Indeed when a method incorporates this type of degree of freedom the definition of isoparametric finite-element analogs is not always simple or clear (see e.g. [3]).
It is important to point out that efficient finite-element techniques are known since long, to optimally handle boundary conditions other than Dirichlet's, such as Neumann or Robin boundary conditions prescribed on curved boundaries. In this respect the author refers for instance to the works by Barrett and Elliott [2] and [1], besides [20] where a clear explanation on the issues brought about by Neumann conditions prescribed on curved boundaries is given.

The technique applied in this paper was introduced in [12] in connection with triangular Lagrange finite elements of any order k greater than one to solve the Poisson equation posed in a smooth curvilinear domain. In the subsequent work [13] the author addressed the case of tetrahedral Lagrange elements of arbitrary order for second order elliptic PDEs in the same class of domains. A synthesis of both papers is given in [15]. In [11] the author and co-worker used the same approach to the solution of Maxwell's equations with a Hermite finite element method. In this work we push further the study of this methodology given in [16] as applied to incompressible viscous flow. More precisely thsi work can be outlined as follows.
Following a brief description of our technique to handle velocity Dirichet conditions on curvilinear boundaries, a priori error estimates in natural norms given [16] for the Taylor-Hood element [8] are recalled. Next such results are extended to our version of the Crouzeix-Raviart triangle [7] for curved domains. Then unprecedented velocity error estimates in the L^2-norm are given for both methods. Numerical results obtained with the second-order Taylor-Hood triangular element and the aforementioned Crouzeix-Raviart method are supplied. A comparison of both methods with their respective isoparametric versions is also carried out.

2 Method description for a model problem

For the sake of conciseness in this work only the two-dimensional case is considered. Applications and studies of the same technique for three-dimensional problems can be found in [13] and [16].

Referring to [12] and [16] for further details, here we describe our technique to solve incompressible flow problems with Dirichlet velocity conditions prescribed on smooth curved boundaries, by solving a simple model problem as follows. Let Ω be a bounded two-dimensional domain and Γ be its boundary with outer normal vector $\mathbf{n}$. Γ is assumed to be sufficiently smooth for the required regularity of the solution to our model problem to hold true. In any case Γ must be of the C^m-class for some $m > 0$.

Let $k > 1$ and $\mathbf{f}$ be given in $[H^{k-1}(\Omega)]^2$. Denoting by $L^2_0(D)$ the subspace of $L^2(D)$ consisting functions whose integral in D vanishes for any bounded open set $D \subset \Re^2$. We wish to find a velocity $\mathbf{u} \in [H^{k+1}(\Omega)]^2$ and a pressure $p \in H^k(\Omega) \cap L^2_0(\Omega)$ that solves the following Stokes system in suitable dimensionless form :

$$\begin{cases} -\Delta \mathbf{u} + \nabla p = \mathbf{f} \text{ in } \Omega \\ \nabla \cdot \mathbf{u} = 0 \text{ in } \Omega \\ \mathbf{u} = \mathbf{g} \text{ on } \Gamma, \end{cases} \tag{1}$$

assuming that $\mathbf{g}$ satisfies $\oint_\Gamma \mathbf{g} \cdot \mathbf{n}\, ds = 0$ and $\mathbf{g} \in [H^{k+1/2}(\Gamma)]^2$.

Now let $\mathcal{P} = \{\mathcal{T}_h\}_h$ be a uniformly regular family of finite element meshes consisting of straight-edged triangles satisfying the usual compatibility conditions (see e.g. [6]). Every element of $\mathcal{T}_h$ is to be viewed as a closed set. Moreover each one of these meshes is assumed to fit Ω in such a way that all the vertexes of the polygon $\cup_{T \in \mathcal{T}_h} T$ lie on Γ. We denote the interior of this union set by Ω_h. Let Γ_h be the boundary of Ω_h, h_T be the diameter of $T \in \mathcal{T}_h$ and $h := \max_{T \in \mathcal{T}_h} h_T$.

We make the very reasonable assumption that every mesh triangle has no more than one edge in Γ_h. The subset of $\mathcal{T}_h$ consisting of elements having at least one edge on Γ_h is denoted by $\mathcal{S}_h$.

The finite-element approximate problem for (1) will be defined in connection with three linear manifolds or spaces $\mathbf{W}_h$, $\mathbf{V}_h$ and Q_h associated with the mesh $\mathcal{T}_h$, playing the following roles :

$\mathbf{W}_h$ is the set in which the velocity is searched for;

$\mathbf{V}_h$ is the velocity test-function space;

Q_h is the pressure shape- and test-function space.

Now $\tilde{\mathbf{f}} \in [L^2(\Omega_h)]^2$ being the extension of $\mathbf{f}$ by zero to $\Omega_h \setminus \Omega$, we set for $\mathbf{u}, \mathbf{v} \in [H^1(\Omega_h)]^2$ and $q \in L^2(\Omega_h)$:

$$\begin{cases} c_h(u, v) := \int_{\Omega_h} \nabla \mathbf{u} \cdot \nabla \mathbf{v}\, d\mathbf{x} \\ b_h(q, \mathbf{v}) := \int_{\Omega_h} q \nabla \cdot \mathbf{v}\, d\mathbf{x} \\ \text{and } L_h(v) := \int_{\Omega_h} \tilde{\mathbf{f}} \cdot \mathbf{v}\, d\mathbf{x}. \end{cases} \tag{2}$$

Then the finite-element counterpart of (1) is defined in the form of the following system :

$$\begin{cases} \text{Find } \mathbf{u}_h \in \mathbf{W}_h \text{ and } p_h \in Q_h \text{ such that :} \\ c_h(\mathbf{u}_h, \mathbf{v}) + b_h(\mathbf{v}, p_h) = \mathbf{L}_h(\mathbf{v})\ \forall \mathbf{v} \in \mathbf{V}_h, \\ b_h(\mathbf{u}_h, q) = 0\ \forall q \in Q_h. \end{cases} \tag{3}$$

Now let V_h^k be the standard Lagrange finite-element space consisting of continuous functions that vanish on Γ_h, whose restriction to each $T \in \mathcal{T}_h$ belongs to P_k, where P_k is the space of polynomials of degree

less than or equal to k. To make ideas clear, and without loss of essential aspects, let us consider the case where $\mathbf{g} \equiv \vec{0}$.
In order to recover optimal convergence rates in the natural norm for (1), when the problem is defined in a polygon as predicted in the specialized literature (cf. [5]), we propose the following :
Let W_h^k be a space consisting of functions defined in Ω_h whose restriction to every element $T \in \mathcal{T}_h$ belongs to P_k, which are continuous at the vertexes of T and at the same (Lagrange) nodes used to define functions in V_h^k not located on the edges of T contained in Γ_h, $\forall T \in \mathcal{T}_h$. Besides this, a function $w \in W_h^k$ is required to vanish at all the vertexes of Γ_h, and at certain points $P \in \Gamma$ belonging to the set Δ_T attached to an elements $T \in \mathcal{S}_h$ containing the underlying portion of Γ, constructed as described below $\forall T \in \mathcal{S}_h$.
Δ_T is the closed set delimited by Γ and the edge e_T of T contained in Γ_h. For every $T \in \mathcal{S}_h$ the extension of w to $\Delta_T \setminus \Omega_h$ is required to vanish at $k-1$ points $P \in \Gamma$ located between two neighboring vertexes of Γ_h. To make implementation more straightforward every such a node P can be chosen to be the nearest intersection with Γ of the line joining the vertex O_T of T opposite to e_T to a node M located on e_T used to define functions in V_h.
We consider beforehand that the expression of $w \in W_h$ in every element $T \in \mathcal{S}_h$ is extended to $\Delta_T \setminus \Omega_h$.
In Figure 1 we illustrate the construction of nodes $P \in \Gamma$ for $k = 3$.
Before going into the specific finite-element analogs of (1) studied in this paper we give some further definitions. $\tilde{\Omega}_h$ being the set $\Omega \cap \Omega_h$ we shall denote by $\| \cdot \|_{0,h}$ the standard norm of $L^2(\tilde{\Omega}_h)$, and by $|\mathbf{w}|_{1,h}$ the semi-norm $\| \nabla \mathbf{w} \|_{0,h}$ of a field $\mathbf{w} \in [H^1(\tilde{\Omega}_h)]^2$. Notice that in case Ω is convex $\tilde{\Omega}_h$ is nothing but Ω_h.

2.1 Approximation by the Taylor-Hood element

In the case of the second-order Taylor-Hood element we work with the following discrete spaces :

$$\mathbf{W}_h := [W_h^2]^2;$$

$$\mathbf{V}_h := [V_h^2]^2;$$

$$Q_h := V_h^1 \cap L_0^2(\Omega_h).$$

For such choices the following error estimate was proven in [16] :

Theorem 1 *Set $\tilde{\Omega} := \Omega_h \cup \Omega$ for all $\mathcal{T}_h \in \mathcal{P}$ and assume that there exists $\tilde{\mathbf{u}}$ and $\tilde{p}$ defined in $\tilde{\Omega}$ fulfilling :*

$$\tilde{\mathbf{u}}_{|\Omega} = \mathbf{u} \textit{ and } \tilde{p}_{|\Omega} = p;$$

$$\tilde{\mathbf{u}} = \overrightarrow{0} \textit{ a.e. on } \Gamma;$$

$$\tilde{\mathbf{u}} \in [H^3(\tilde{\Omega}_h)]^2 \textit{ and } \tilde{p} \in H^2(\tilde{\Omega}_h).$$

Then as long as h is small enough, for a certain mesh-independent constant $\tilde{\mathcal{C}}(\tilde{\mathbf{u}}, \tilde{p})$ it holds :

$$[|\mathbf{u} - \mathbf{u}_h|_{1,h}^2 + \| p - p_h \|_{0_h}^2]^{1/2} \leq \tilde{\mathcal{C}}(\tilde{\mathbf{u}}, \tilde{p}) h^2. \blacksquare \tag{4}$$

Remark 1 *The construction of $\tilde{\mathbf{u}}$ and $\tilde{p}$ can be accomplished in many ways, for example as advocated in [19].* ■

2.2 Approximation by the conforming Crouzeix-Raviart element

In this case the discrete spaces are defined as follows.
For the pressure we first introduce the space

$$R_h := \{q|\ q_{|T} \in P_1\ \forall T \in \mathcal{T}_h\}.$$

As for the velocity we first denote by φ_T the cubic (bubble) function that vanishes on the three edges of $T \in \mathcal{T}$, extended by zero outside T. Further T being a triangle in $\mathcal{S}_h$ we denote by ψ_T the cubic function defined in T that vanishes on the two edges of T not lying on Γ_h and whose extension to Δ_T and beyond also vanishes along the line parallel to e_T passing through node $P \in \Delta_T \cap \Gamma$ of the space W_h^2. We set ψ_T identically zero in every $T \in \mathcal{T}_h \setminus \mathcal{S}_h$.
Now we introduce the spaces,

$$\Psi_h := \{\psi|\ \psi_{|T} = span[\psi_T]\ \forall T \in \mathcal{S}_h\},$$

$$\Phi_h := \{\varphi|\ \varphi_{|T} = span[\varphi_T]\ \forall T \in \mathcal{T}_h\} \text{ and}$$

$$\tilde{\Phi}_h := \{\varphi|\ \varphi_{|T} = span[\varphi_T]\ \forall T \in \mathcal{T}_h \setminus \mathcal{S}_h\}.$$

The finite-element analog (3) of (1) for the conforming Crouzeix-Raviart element in the case of curved domains is defined by means of the following choice of discrete spaces :

$$\mathbf{W}_h := [W_h^2 \oplus \Psi_h \oplus \tilde{\Phi}_h]^2;$$

$$\mathbf{V}_h := [V_h^2 \oplus \Phi_h]^2;$$

$$Q_h := R_h \cap L_0^2(\Omega_h).$$

Error estimates qualitatively equivalent to (4) hold for the resulting approximate problem under the same assumptions as in Theorem 1.

2.3 Mean-square error estimates for the velocity

If we require a little more regularity from both the velocity and the pressure we can assert that an error estimate in terms of h^3 holds for the velocity in the norm $\| \cdot \|_{0,h}$. More precisely such an estimate is obtained under the assumption that $\mathbf{u} \in [H^{3.5+\epsilon}(\Omega)]^2$ and $p \in H^{2.5+\epsilon}(\Omega)$ where ϵ is a strictly positive real number arbitrarily small. The proof of this error estimate is a rather straightforward variant of the very lengthy proof of an equivalent estimate for the scalar Poisson problem given in [12]. Details will be supplied in a forthcoming paper. In this work we confine ourselves to their numerical validation provided in the next section among other experiments.

3 Numerical experiments

The primary aim of this section is the numerical validation of the theoretical predictions of the previous section for two well established mixed methods to solve the incompressible Navier-Stokes equations combined with our technique to handle Dirichlet velocity boundary conditions prescribed on curvilinear boundaries. We also compare it with their corresponding parametric versions in terms of accuracy. In

order to enable a fair comparison we solved only toy problems governed by the Stokes system (1) with known exact solution.
As far as the Taylor-Hood method is concerned most of the numerical results given in this section are borrowed from [16].
In our computations the pressure vanishes at a given point and do not satisfy the zero integral condition.
In the tables that follow the acronym OCR stands for *observed convergence rate*.

3.1 Test-problem 1

To begin with we solve (1) with a manufactured solution corresponding to the following data : Ω is the unit disk (centered at the origin), $\mathbf{f} = (8,8)(x-y)$, and $\mathbf{g} \equiv \overrightarrow{0}$. Prescribing $p(\sqrt{2}/2, \sqrt{2}/2) = 0$, the exact solution has polynomial expressions, namely $\mathbf{u} = (y, -x)(1 - x^2 - y^2)$ and $p = x^2 - y^2$.
We use a quasi-uniform family of meshes constructed as prescribed in [16] for the whole disk, parametrized by an integer n, consisting of $8n^2$ triangles, each mesh being symmetric with respect to the axes $x = 0$ and $y = 0$, for $n = 2^m$ with $m = 1, 2, 3, 4$. In this way we can take $h = 1/n$.
In the tables below the notations $\tilde{\mathbf{u}}_h$ and $\tilde{p}_h$ are employed to represent the velocity and pressure obtained by the quadratic parametric approach for the velocity, which is isoparametric in the case of the Taylor-Hood method and subparametric in the case of the Crouzeix-Raviart method. We display in Tables 1 and 2 the velocity errors in the norms $|\cdot|_{1,h}$ and $|\cdot|_{0,h}$ and the pressure errors in the norm $\|\cdot\|_{0,h}$ for the Taylor-Hood and the Crouzeix-Raviart method, respectively, combined with our technique and the quadratic parametric approach for the velocity. As one can see, these results completely validate the analysis carried out in the previous sections.

As one can observe from Tables 1 and 2 the new approach is more accurate than the parametric approach in all respects.

Remark 2 *It is noteworthy that according to Tables 1 and 2, the Taylor-Hood method appears to be roughly as accurate as the Crouzeix-Raviart method, as far as the velocity is concerned. On the other hand the pressure is significantly better approximated with the former for finer meshes. This fact is in contradiction with observations made by other authors (see e.g. [9]) for rectangular domains. However we should clarify that the bubble functions were not taken into account in the computation of velocity errors, which certainly increases them, though keeping the orders intact. Nevertheless since this procedure does not affect the pressure our experiments seem to indicate that for curved domains the Taylor-Hood method tend to perform better than the Crouzeix-Raviart method, as far as this field is concerned. But of course much more experimentation is necessary to confirm or not such a conclusion.* ■

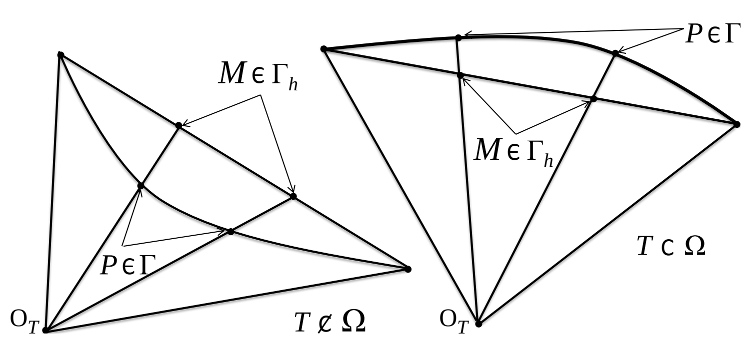

FIGURE 1 – Construction of nodes $P \in \Gamma$ for space W_h related to Lagrange nodes $M \in \Gamma_h$ for $k = 3$

TABLE 1 – Errors for the Taylor-Hood method combined with the new and the parametric approach

$2n$	$\longrightarrow$	4	8	16	32	OCR
$\|\mathbf{u}_h - \mathbf{u}\|_{1,h}$	$\longrightarrow$	0.1308×10^{-0}	0.3585×10^{-1}	0.8833×10^{-2}	0.2157×10^{-2}	$O(h^2)$
$\|\tilde{\mathbf{u}}_h - \mathbf{u}\|_{1,h}$	$\longrightarrow$	0.1554×10^{-0}	0.3938×10^{-1}	0.9321×10^{-2}	0.2222×10^{-2}	$O(h^2)$
$\|\mathbf{u}_h - \mathbf{u}\|_{0,h}$	$\longrightarrow$	0.8633×10^{-2}	0.1211×10^{-2}	0.1446×10^{-3}	0.1717×10^{-4}	$O(h^3)$
$\|\tilde{\mathbf{u}}_h - \mathbf{u}\|_{0,h}$	$\longrightarrow$	0.9726×10^{-2}	0.1308×10^{-2}	0.1522×10^{-3}	0.1770×10^{-4}	$O(h^3)$
$\parallel p_h - p \parallel_{0,h}$	$\longrightarrow$	0.1709×10^{-0}	0.4266×10^{-1}	0.1047×10^{-1}	0.2567×10^{-2}	$O(h^2)$
$\parallel \tilde{p}_h - p \parallel_{0,h}$	$\longrightarrow$	0.1920×10^{-0}	0.4508×10^{-1}	0.1077×10^{-1}	0.2604×10^{-2}	$O(h^2)$

TABLE 2 – Errors for the Crouzeix-Raviart method combined with the new and the parametric approach

$2n$	$\longrightarrow$	8	16	32	64	OCR
$\|\mathbf{u}_h - \mathbf{u}\|_{1,h}$	$\longrightarrow$	0.1238×10^{-0}	0.3292×10^{-1}	0.8659×10^{-2}	0.2216×10^{-2}	$O(h^2)$
$\|\tilde{\mathbf{u}}_h - \mathbf{u}\|_{1,h}$	$\longrightarrow$	0.1515×10^{-0}	0.3666×10^{-1}	0.9152×10^{-2}	0.2280×10^{-2}	$O(h^2)$
$\|\mathbf{u}_h - \mathbf{u}\|_{0,h}$	$\longrightarrow$	0.8567×10^{-2}	0.1056×10^{-2}	0.1360×10^{-3}	0.1732×10^{-4}	$O(h^3)$
$\|\tilde{\mathbf{u}}_h - \mathbf{u}\|_{0,h}$	$\longrightarrow$	0.1047×10^{-1}	0.1201×10^{-2}	0.1458×10^{-3}	0.1796×10^{-4}	$O(h^3)$
$\parallel p_h - p \parallel_{0,h}$	$\longrightarrow$	0.1675×10^{-0}	0.4048×10^{-1}	0.1072×10^{-1}	0.3182×10^{-2}	$O(h^{\approx 2})$
$\parallel \tilde{p}_h - p \parallel_{0,h}$	$\longrightarrow$	0.1825×10^{-0}	0.4294×10^{-1}	0.1109×10^{-1}	0.3232×10^{-2}	$O(h^{\approx 2})$

3.2 Test-problem 2

In order to check our method's performance in a more physical context we used it to solve a problem related to circular Couette flow of a viscous incompressible fluid with density ρ, in a region comprised between two concentric cylinders. The inner cylinder of radius r_i rotates with angular velocity ω while the outer cylinder with radius r_e is kept fixed. This flow is governed by the stationary Navier-Stokes equations with a zero body-force right hand side. As long as the Reynolds number is sufficiently low, the flow is laminar and the solution to the problem is given by $\mathbf{u} = (sin\theta, -cos\theta)u_\theta(r)$ where $u_\theta(r) = \omega r_i^2(r_e^2 - r^2)/[r(re^2 - ri^2)]$ and $p(r) = \rho\omega^2 r_i^4/(r_e^2 - r_i^2)^2[r^2/2 - r_e^4/(2r^2) - 2r_e^2 log(r)] + c$, c being a constant. If we enforce zero pressure on the outer wall, then c takes the value $2r_e^2 log(r_e)\rho\omega^2 r_i^4/(r_e^2 - r_i^2)^2$.
Although there is no particular difficulty to solve the Navier-Stokes equations with our method, in order to focus on our essentially validating goal, we apply it to a modified problem, in which the exact inertia term $\rho[\mathbf{grad}\ \mathbf{u}]\mathbf{u}$ with a minus sign is input as right hand side datum $\mathbf{f}$. Of course the pair $(\mathbf{u}, p)$ is still the solution to the resulting Stokes system (1) in the annulus Ω with inner radius r_i and outer radius r_e. The datum $\mathbf{g}$ in turn equals $\overrightarrow{0}$ for $r = r_e$, while its value for $r = r_i$ conforms to the given azimuthal velocity $r_i\omega$ and a zero radial velocity.
Taking $r_e = 1$, $r_i = 0.5$, $\omega = 1$ and $\rho = 1$, we proceeded to the numerical solution of thus defined

(pseudo) circular Couette flow problem with the Taylor-Hood and the Crouzeix-Raviart method combined with our technique to approximate the boundary conditions. In order to avoid non physical boundary conditions, computations were carried out for the whole annulus. With this aim we used again $(2n \times 2n)$ symmetric meshes, for $n = 2^m$, with $m = 2, 3, 4, 5$, constructed in the same way as in the previous subsection, except for the fact that now the elements inside the disk with radius r_i were disregarded. This yields meshes consisting of $6n^2$ triangles, with $h = 1/n$. At numerical level a zero pressure value is enforced at the point$(\sqrt{2}/2, \sqrt{2}/2)$ for both methods.
We display in Table 3 (resp. 4) the velocity errors measured in the norms $|\cdot|_{1,h}$ and $\|\cdot\|_{0,h}$, together with the pressure errors measured in the $\|\cdot\|_{0,h}$-norm, for the Taylor-Hood (resp. Crouzeix-Raviart) method in formulation (3). Results with the parametric technique are commented but not exhibited.
From both tables we observe that the velocity errors in the H^1-semi-norm and in the L^2-norm are in perfect agreement with the theoretical predictions. Moreover here the Taylor-Hood element behaves undoubtedly better than the Crouzeix-Raviart element, at least when the bubble functions are neglected in the error computations. On the other hand we note that the pressure errors for the Taylor-Hood method decrease at a rate faster than the $O(h^2)$ observed for the Test-problem 1. However for the Crouzeix-Raviart method the latter errors decrease only at a rate close to $O(h^{3/2})$. This reveals an amplification of the effects pointed out in Remark 2 with respect to pressure errors, in the framework of flow simulations in curved domains. In principle such results contradict our theoretical predictions for the Crouzeix-Raviart method. Thus the author intends to further investigate this issue in future work. Incidentally such a pressure downgrade effect is also observed for the parametric version of the Crouzeix-Raviart method, while the aforementioned pressure upgrade effect for the Taylor-Hood element does not occur in computations using the isoparametric technique. Taking into account that in this example too the approximation of the velocity with the new procedure is of comparable accuracy with its parametric approximation for both methods, we can assert that here again the former showed to be superior to the latter.

TABLE 3 – Errors for the Taylor-Hood method combined with formulation (3)

$2n$	$\longrightarrow$	8	16	32	64	OCR
$\lvert\mathbf{u}_h - \mathbf{u}\rvert_{1,h}$	$\longrightarrow$	0.1592×10^{-0}	0.4261×10^{-1}	0.1090×10^{-1}	0.2741×10^{-2}	$O(h^2)$
$\lVert \mathbf{u}_h - \mathbf{u} \rVert_{0,h}$	$\longrightarrow$	0.3833×10^{-2}	0.5339×10^{-3}	0.6923×10^{-4}	0.8744×10^{-5}	$O(h^3)$
$\lVert p_h - p \rVert_{0,h}$	$\longrightarrow$	0.1209×10^{-0}	0.2095×10^{-1}	0.3952×10^{-2}	0.6638×10^{-3}	$O(h^{\approx 2.5})$

TABLE 4 – Errors for the Crouzeix-Raviart method combined with formulation (3)

$2n$	$\longrightarrow$	8	16	32	64	OCR
$\lvert\mathbf{u}_h - \mathbf{u}\rvert_{1,h}$	$\longrightarrow$	0.1903×10^{-0}	0.5433×10^{-1}	0.1431×10^{-1}	0.3508×10^{-2}	$O(h^2)$
$\lVert \mathbf{u}_h - \mathbf{u} \rVert_{0,h}$	$\longrightarrow$	0.6372×10^{-2}	0.9448×10^{-3}	0.1229×10^{-3}	0.1456×10^{-4}	$O(h^3)$
$\lVert p_h - p \rVert_{0,h}$	$\longrightarrow$	0.2814×10^{-0}	0.1177×10^{-0}	0.4824×10^{-1}	0.1727×10^{-1}	$O(h^{\approx 1.5})$

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