Upscaling of two-phase flow with chemical reactions in double porosity media

Alexandre Koldoba[a], Andrey Konyukhov[a,b], and Leonid Pankratov[a,c]

a. Laboratory of Fluid Dynamics and Seismic, Moscow Institute of Physics and Technology, 9 Institutskiy per., Dolgoprudny, Moscow Region, 141700, Russian Federation. koldoba@rambler.ru
b. Joint Institute for High Temperatures of the Russian Academy of Sciences, Izhorskaya 13 Bldg, 2, Moscow, 125412, Russian Federation. konyukhov_av@mail.ru
c. Laboratoire de Mathématiques et de leurs Applications-IPRA, CNRS/UNIV PAU & PAYS ADOUR, UMR 5142, Av. de l'Université, 64000 Pau, France. leonid.pankratov@univ-pau.fr

Abstract :

The paper deals with homogenization of reactive immiscible incompressible two-phase flow in double porosity media. The mathematical model is given by a coupled system of two-phase flow equations under isothermal condition. The model consists of the usual equations derived from the mass conservation of both fluids along with the Darcy-Muskat and the capillary pressure laws along with the additional source terms corresponding to the chemical reactions in the reservoir. The medium is made of two superimposed continua, a connected fracture system and an ε-periodic system of disjoint matrix blocks. We assume that the permeability of the fissures is of order one, while the permeability of the blocks is of order ε^2. We derive the global behavior of the model by passing to the limit as $\varepsilon \to 0$ and obtain the global model of the reactive flow. It is shown that the homogenized model can be represented as the usual equations of a reactive immiscible incompressible two-phase flow except for the addition of new source terms calculated by a solution to a local problem in the matrix block. These source terms exhibit the nonlocal behavior of the model with respect to the time variable. The non-locality in time of the reaction source terms in the case of gas producing reaction can lead to the instability of stationary reaction front propagation in the porous medium and development of self-oscillations. The results of the numerical simulation of the reactive immiscible incompressible two-phase flow are presented.

Key words : two-phase flow ; chemical reactions ; homogenization ; double porosity media.

1 Introduction

The homogenization of reactive two-phase flows in porous media is of great interest in view of numerous applications in geosciences and technologies (see, e.g., [17]). There is an extensive literature on this subject. We will not attempt a literature review here but will merely mention a few references. Here we refer, for instance, to [13, 16] and the references therein.

In this paper we consider a more general model than the previous ones. It describes a two-phase flow in double porosity media with chemically active skeleton. The chemical reaction between a solid reactant

and one dissolved in the wetting liquid is accompanied by production of non-wetting (gas) phase. Naturally fractured reservoirs can be modeled by two-superimposed continua, a connected fracture system and a system of topologically disconnected matrix blocks. The fracture system has a low storage capacity and high a conductivity, while the matrix block system has a conductivity that is low in comparison to that in the fractures. The majority of fluid transport will occur along flow paths through the fissure system, and the relative volume and storage capacity of the porous matrix is much larger than that of the fissure system. When the system of fissures is so well developed that the matrix is broken into individual blocks or cells that are isolated from each other, there is consequently no flow directly from cell to cell, but only an exchange of fluid between each cell and the surrounding fissure system. For more details on the physical formulation of such problems see, e.g. [21, 25].

The large-scale description will have to incorporate the two different flow mechanisms. For some permeability ratios and some fissures width, the large-scale description is achieved by introducing the so-called double-porosity model. It was introduced first for describing the global behavior of fractured porous media by Barenblatt et al. [6] and it is since used in a wide range of engineering specialties related to geohydrology, petroleum reservoir engineering, civil engineering or soil science.

The usual double-porosity model (or ε^2-model) assumes that the width of the fracture containing highly permeable porous media is of the same order as the block size. The related homogenization problem was first studied in [4], and was then revisited in the mathematical literature by many other authors. After a series of papers, the notion of double-porosity media was associated mainly to ε^2-model. The appearance of the additional source terms in the homogenized problems is studied in [5]. In this paper we make use of this normalization. More general notion of double-porosity media was introduced in [21], where the medium was called as double-porosity when $r = r(\varepsilon)$, the ratio between the permeability of the matrix blocks and the fissures system is such that $0 < r < 1$. The ε-periodic double porosity medium considered in this paper consists of disjoint low permeable matrix blocks and a connected highly permeable fracture system. The homogenization process is done by passing to the limit as $\varepsilon \to 0$. The global (homogenized) model contains the nonlocal in time additional source terms. Notice that recently, in the case of two-phase non-reacting flows was studied, for instance, in [1–3, 18, 19, 22, 23] (see also the references therein).

One of the goals of this paper (in addition to the rigorous justification of the double porosity model) is to emphasize the connection between the non-locality of sources associated with the response and the stability of solutions of the initial boundary value problems. Indeed, from a physical point of view, the manifestation of such non-locality for a system with double porosity is due to the presence of a time delay between the change in the averaged concentrations of the reactants and the averaged reaction rate. This is typical for the case when the convection of the reagent is carried out in fractures, and the reaction takes place in the matrix. In the fluxes where the flow of a reactant into the reaction zone depends on the reaction rate (for example, a reaction with evolution of a gas phase, leading to an increase in pore pressure), this time delay can lead to the presence of bifurcations of the solution. A mathematical example is the equation with a retarded argument. Here we can note an experiment on the propagation of the reaction front during the injection of a reagent into a porous medium with a chemically active skeleton [26]. In this experiment, the instability of the quasi-stationary propagation of the reaction front and the development of self-oscillations were observed.

The rest of the paper is organized as follows. In Section 2 a mesoscopic flow model is introduced. Then in Section 3 we are dealing with the upscaling of the two-phase flow model given in the previous sec-

tion. The upscaling process is done in the framework of two-scale asymptotic expansions formalism. The resulting homogenized model is represented as the usual equations of a reactive immiscible incompressible two-phase flow except for the addition of new source terms calculated by a solution to a local problem in the matrix block. These source terms exhibit the nonlocal behavior of the model with respect to the time variable. In Section 4 the reaction wave propagation accompanied by the production of the gas phase in the double porosity layer with chemically active matrix is considered. The attention is focused on the influence of the non-local in time source terms on instability of stationary mode of reaction front propagation and transition to the self-oscillating mode of the flow under consideration. Lastly, some concluding remarks are forwarded.

2 Mesoscopic flow model and main assumptions

In this section we formulate the mesoscopic flow equations of the model. We consider a reservoir $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ which is assumed to be a bounded, connected domain with a periodic structure. More precisely, we will scale this periodic structure by a parameter ε which represents the ratio of the cell size to the whole region Ω and we assume that $\varepsilon \downarrow 0$. Let $Y \stackrel{\text{def}}{=} (0,1)^d$ be a basic cell of a fractured porous medium. We assume that Y is made up of two homogeneous porous media Y_{m} and Y_{f} corresponding to the parties of the mesoscopic domain occupied by the matrix block and the fracture, respectively. Thus $Y = Y_{\mathsf{m}} \cup Y_{\mathsf{f}} \cup \Gamma_{\mathsf{fm}}$, where Γ_{fm} denotes the interface between the two media. Let Ω_ℓ^ε with $\ell =$ "f" or "m" denotes the open set corresponding to the porous medium with index ℓ. Then $\Omega = \Omega_{\mathsf{m}}^\varepsilon \cup \Gamma_{\mathsf{fm}}^\varepsilon \cup \Omega_{\mathsf{f}}^\varepsilon$, where $\Gamma_{\mathsf{fm}}^\varepsilon \stackrel{\text{def}}{=} \partial\Omega_{\mathsf{f}}^\varepsilon \cap \partial\Omega_{\mathsf{m}}^\varepsilon \cap \Omega$ and the subscripts "m", "f" refer to the matrix and fracture, respectively. For the sake of simplicity, we assume that $\Omega_{\mathsf{m}}^\varepsilon \cap \partial\Omega = \emptyset$. We also introduce the notation :

$$\Omega_T \stackrel{\text{def}}{=} \Omega \times (0,T), \quad \Omega_{\ell,T}^\varepsilon \stackrel{\text{def}}{=} \Omega_\ell^\varepsilon \times (0,T), \quad \Sigma_T^\varepsilon \stackrel{\text{def}}{=} \Gamma_{\mathsf{fm}}^\varepsilon \times (0,T), \quad \text{where } T > 0 \text{ is fixed.} \tag{1}$$

We notice here that in our starting mesoscopic model the fractures are represented as a porous medium with rock properties radically different from those of the matrix blocks. In particular, they are not represented as an empty space filled with the fluids. For an example of a numerical simulation over 3D matrix-fracture structure described here we refer to [12].

Before describing the equations of the model for the nonhomogeneous porous medium Ω with the periodic microstructure, we give the notation for the physical quantities used in the paper. We also define the porosity function and the global permeability tensor adopted to the fractured-porous medium Ω with double porosity. We have :

– $\Phi^\varepsilon(x) = \Phi(\frac{x}{\varepsilon})$ be the porosity of the reservoir Ω. The function Φ^ε is a Y-periodic defined by :

$$\Phi^\varepsilon(x) \stackrel{\text{def}}{=} \begin{cases} \Phi_{\mathsf{f}} & \text{in } \Omega_{\mathsf{f}}^\varepsilon; \\ \Phi_{\mathsf{m}} & \text{in } \Omega_{\mathsf{m}}^\varepsilon, \end{cases} \tag{2}$$

where the constants $0 < \Phi_{\mathsf{f}}, \Phi_{\mathsf{m}} < 1$ do not depend on ε.

– $K^\varepsilon(x) = K(\frac{x}{\varepsilon})$ be the absolute permeability tensor of Ω. The function K^ε is defined by :

$$K^\varepsilon(x) \stackrel{\text{def}}{=} \begin{cases} K_{\mathsf{f}}\,\mathbb{I} & \text{in } \Omega_{\mathsf{f}}^\varepsilon; \\ \varepsilon^2 K_{\mathsf{m}}\,\mathbb{I} & \text{in } \Omega_{\mathsf{m}}^\varepsilon, \end{cases} \tag{3}$$

where $\mathbb{I}$ is the unit tensor and K_{f}, K_{m} are positive constants that do not depend on ε.

- $S^\varepsilon_{\ell,w} = S^\varepsilon_{\ell,w}(x,t)$, $S^\varepsilon_{\ell,n} = S^\varepsilon_{\ell,n}(x,t)$ are the saturations of wetting and nonwetting fluids in the medium Ω^ε_ℓ, respectively ($\ell = \mathsf{f}, \mathsf{m}$).
- $p^\varepsilon_{\ell,w} = p^\varepsilon_{\ell,w}(x,t)$, $p^\varepsilon_{\ell,n} = p^\varepsilon_{\ell,n}(x,t)$ are the pressures of wetting and nonwetting fluids in the medium Ω^ε_ℓ, respectively ($\ell = \mathsf{f}, \mathsf{m}$).
- $f^{(\ell)}_w = f^{(\ell)}_w(S^\varepsilon_{\ell,w})$, $f^{(\ell)}_n = f^{(\ell)}_n(S^\varepsilon_{\ell,n})$ are the relative permeabilities of wetting and nonwetting fluids in the medium Ω^ε_ℓ, respectively ($\ell = \mathsf{f}, \mathsf{m}$).
- $\mathsf{c}^\varepsilon_\ell = \mathsf{c}^\varepsilon_\ell(x,t)$ ($0 \leqslant \mathsf{c}^\varepsilon_\ell \leqslant 1$) is the reagent concentration in the wetting phase ($\ell = \mathsf{f}, \mathsf{m}$).
- $\mathsf{M}^\varepsilon_\ell = \mathsf{M}^\varepsilon_\ell(x,t)$ is the averaged density of the solid reagent.
- $\mathcal{D}^\varepsilon_\ell = \mathcal{D}_\ell(S^\varepsilon_{\ell,w})$ is the diffusion coefficient.
- μ_w, μ_n are the viscosities of the wetting and nonwetting fluids, respectively.
- $P_{\ell,c}(S^\varepsilon_{\ell,w})$ is the capillary pressure function given by :

$$P_{\ell,c}(S^\varepsilon_{\ell,w}) \overset{\text{def}}{=} p^\varepsilon_{\ell,n} - p^\varepsilon_{\ell,w} \quad \text{with } P'_{\ell,c}(s) < 0 \text{ for all } s \in [0,1] \text{ and } P_{\ell,c}(1) = 0, \tag{4}$$

  where $P'_{\ell,c}(s)$ denotes the derivative of the function $P_{\ell,c}(s)$.

Now we are in position to introduce the flow equations. For the sake of simplicity and brevity, we neglect the gravity phenomena. The model consists of the usual equations derived from the mass conservation of both fluids along with the Darcy-Muskat (see, e.g., [7, 10, 11, 15]) and the capillary pressure laws. The problem is written in terms of the phase formulation, i.e., the saturations, the pressures, and the concentrations are the primary unknowns.

Mass Balance in Fractures. The flow equations in the fracture system $\Omega^\varepsilon_{\mathsf{f},T}$ read :

$$\begin{cases} \Phi^\varepsilon(x)\dfrac{\partial S^\varepsilon_\mathsf{f}}{\partial t} - \operatorname{div}\big\{K_\mathsf{f}\lambda_{\mathsf{f},w}(S^\varepsilon_\mathsf{f})\nabla p^\varepsilon_{\mathsf{f},w}\big\} = \mathbb{S}^\varepsilon_{\mathsf{f},w}; \\ -\Phi^\varepsilon(x)\dfrac{\partial S^\varepsilon_\mathsf{f}}{\partial t} - \operatorname{div}\big\{K_\mathsf{f}\lambda_{\mathsf{f},n}(S^\varepsilon_\mathsf{f})\nabla p^\varepsilon_{\mathsf{f},n}\big\} = \mathbb{S}^\varepsilon_{\mathsf{f},n}; \\ \Phi^\varepsilon(x)\dfrac{\partial}{\partial t}\big[S^\varepsilon_\mathsf{f}\,\mathsf{c}^\varepsilon_\mathsf{f}\big] - \operatorname{div}\big\{K_\mathsf{f}\,\mathsf{c}^\varepsilon_\mathsf{f}\,\lambda_{\mathsf{f},w}(S^\varepsilon_\mathsf{f})\nabla p^\varepsilon_{\mathsf{f},w} + \mathcal{D}_\mathsf{f}(S^\varepsilon_\mathsf{f})\nabla \mathsf{c}^\varepsilon_\mathsf{f}\big\} = \mathbb{S}^\varepsilon_{\mathsf{f},w}; \\ P_{\mathsf{f},c}(S^\varepsilon_\mathsf{f}) = p^\varepsilon_{\mathsf{f},n} - p^\varepsilon_{\mathsf{f},w}. \end{cases} \tag{5}$$

Here $S^\varepsilon_\mathsf{f} \overset{\text{def}}{=} S^\varepsilon_{\mathsf{f},w}$ is the wetting fluid saturation in the fissures system ($0 \leqslant S^\varepsilon_\mathsf{f} \leqslant 1$); $\lambda_{\mathsf{f},w}(S^\varepsilon_\mathsf{f})$ and $\lambda_{\mathsf{f},n}(S^\varepsilon_\mathsf{f}) := \lambda_{\mathsf{f},n}(1 - S^\varepsilon_\mathsf{f})$ stand for the mobilities of the wetting and non-wetting phases given by :

$$\lambda_{\mathsf{f},w}(S^\varepsilon_\mathsf{f}) \overset{\text{def}}{=} \frac{f^{(\mathsf{f})}_w}{\mu_w}(S^\varepsilon_\mathsf{f}) \quad \text{and} \quad \lambda_{\mathsf{f},n}(S_\mathsf{f}) \overset{\text{def}}{=} \frac{f^{(\mathsf{f})}_n}{\mu_n}(S^\varepsilon_\mathsf{f}); \tag{6}$$

$$\mathbb{S}^\varepsilon_{\mathsf{f},w} \overset{\text{def}}{=} \mathbb{C}_{\mathsf{f},w}\, S^\varepsilon_\mathsf{f}\, \mathsf{M}^\varepsilon_\mathsf{f}\, \mathsf{c}^\varepsilon_\mathsf{f} \text{ and } \mathbb{S}^\varepsilon_{\mathsf{f},n} \overset{\text{def}}{=} \mathbb{C}_{\mathsf{f},n}\, [1 - S^\varepsilon_\mathsf{f}]\, \mathsf{M}^\varepsilon_\mathsf{f}\, \mathsf{c}^\varepsilon_\mathsf{f}, \tag{7}$$

where $\mathbb{C}_{\mathsf{f},w}, \mathbb{C}_{\mathsf{f},n}$ are strictly positive constants which does not depend on ε ; the averaged density of the solid reagent $\mathsf{M}^\varepsilon_\mathsf{f}$ satisfies the following kinetic equation :

$$\begin{cases} \dfrac{\partial \mathsf{M}^\varepsilon_\mathsf{f}}{\partial t} = \mathbb{S}^\varepsilon_{\mathsf{f},\mathsf{M}} \quad \text{in } \Omega^\varepsilon_{\mathsf{f},T}; \\ \mathsf{M}^\varepsilon_\mathsf{f}(x,0) = 1 \end{cases} \quad \text{with} \quad \mathbb{S}^\varepsilon_{\mathsf{f},\mathsf{M}} \overset{\text{def}}{=} \mathbb{C}_{\mathsf{f},\mathsf{M}}\, S^\varepsilon_\mathsf{f}\, \mathsf{M}^\varepsilon_\mathsf{f}\, \mathsf{c}^\varepsilon_\mathsf{f}, \tag{8}$$

where $\mathbb{C}_{\mathsf{f},\mathsf{M}}$ is a strictly negative constant which does not depend on ε.

Mass Balance in Matrix Blocks. The flow equations in the fracture system $\Omega^{\varepsilon}_{\mathsf{m},T}$ read :

$$\begin{cases} \Phi^{\varepsilon}(x)\dfrac{\partial S^{\varepsilon}_{\mathsf{m}}}{\partial t} - \varepsilon^2 \operatorname{div}\big\{K_{\mathsf{m}}\lambda_{\mathsf{m},w}(S^{\varepsilon}_{\mathsf{m}})\nabla p^{\varepsilon}_{\mathsf{m},w}\big\} = \mathbb{S}^{\varepsilon}_{\mathsf{m},w}; \\ -\Phi^{\varepsilon}(x)\dfrac{\partial S^{\varepsilon}_{\mathsf{m}}}{\partial t} - \varepsilon^2 \operatorname{div}\big\{K_{\mathsf{m}}\lambda_{\mathsf{m},n}(S^{\varepsilon}_{\mathsf{m}})\nabla p^{\varepsilon}_{\mathsf{m},n}\big\} = \mathbb{S}^{\varepsilon}_{\mathsf{m},n}; \\ \Phi^{\varepsilon}(x)\dfrac{\partial}{\partial t}\big[S^{\varepsilon}_{\mathsf{m}}\,\mathsf{c}^{\varepsilon}_{\mathsf{m}}\big] - \varepsilon^2 \operatorname{div}\big\{K_{\mathsf{m}}\,\mathsf{c}^{\varepsilon}_{\mathsf{m}}\,\lambda_{\mathsf{m},w}(S^{\varepsilon}_{\mathsf{m}})\nabla p^{\varepsilon}_{\mathsf{m},w} + \mathcal{D}_{\mathsf{m}}(S^{\varepsilon}_{\mathsf{m}})\nabla \mathsf{c}^{\varepsilon}_{\mathsf{m}}\big\} = \mathbb{S}^{\varepsilon}_{\mathsf{m},w}; \\ P_{\mathsf{m},c}(S^{\varepsilon}_{\mathsf{m}}) = p^{\varepsilon}_{\mathsf{m},n} - p^{\varepsilon}_{\mathsf{m},w}. \end{cases} \tag{9}$$

Here $S^{\varepsilon}_{\mathsf{m}} \overset{\text{def}}{=} S^{\varepsilon}_{\mathsf{m},w}$ is the wetting fluid saturation in the matrix blocks ($0 \leqslant S^{\varepsilon}_{\mathsf{m}} \leqslant 1$); $\lambda_{\mathsf{m},w}(S^{\varepsilon}_{\mathsf{m}})$ and $\lambda_{\mathsf{m},n}(S^{\varepsilon}_{\mathsf{m}}) := \lambda_{\mathsf{m},n}(1 - S^{\varepsilon}_{\mathsf{m}})$ stand for the mobilities of the wetting and non-wetting phases given by :

$$\lambda_{\mathsf{m},w}(S^{\varepsilon}_{\mathsf{m}}) \overset{\text{def}}{=} \frac{f^{(\mathsf{m})}_{w}}{\mu_w}(S^{\varepsilon}_{\mathsf{m}}) \quad \text{and} \quad \lambda_{\mathsf{m},n}(S_{\mathsf{m}}) \overset{\text{def}}{=} \frac{f^{(\mathsf{m})}_{n}}{\mu_n}(S^{\varepsilon}_{\mathsf{m}}); \tag{10}$$

$$\mathbb{S}^{\varepsilon}_{\mathsf{m},w} \overset{\text{def}}{=} \mathbb{C}_{\mathsf{m},w}\, S^{\varepsilon}_{\mathsf{m}}\, \mathsf{M}^{\varepsilon}_{\mathsf{m}}\, \mathsf{c}^{\varepsilon}_{\mathsf{m}} \ \text{ and } \ \mathbb{S}^{\varepsilon}_{\mathsf{m},n} \overset{\text{def}}{=} \mathbb{C}_{\mathsf{m},n}\, [1 - S^{\varepsilon}_{\mathsf{m}}]\, \mathsf{M}^{\varepsilon}_{\mathsf{m}}\, \mathsf{c}^{\varepsilon}_{\mathsf{m}}, \tag{11}$$

where $\mathbb{C}_{\mathsf{m},w}, \mathbb{C}_{\mathsf{m},n}$ are strictly positive constants which does not depend on ε ; the averaged density of the solid reagent $\mathsf{M}^{\varepsilon}_{\mathsf{m}}$ satisfies the following kinetic equation :

$$\begin{cases} \dfrac{\partial \mathsf{M}^{\varepsilon}_{\mathsf{m}}}{\partial t} = \mathbb{S}^{\varepsilon}_{\mathsf{m},\mathsf{M}} \quad \text{in } \Omega^{\varepsilon}_{\mathsf{m},T}; \\ \mathsf{M}^{\varepsilon}_{\mathsf{m}}(x,0) = 1 \end{cases} \qquad \text{with} \quad \mathbb{S}^{\varepsilon}_{\mathsf{m},\mathsf{M}} \overset{\text{def}}{=} \mathbb{C}_{\mathsf{m},\mathsf{M}}\, S^{\varepsilon}_{\mathsf{m}}\, \mathsf{M}^{\varepsilon}_{\mathsf{m}}\, \mathsf{c}^{\varepsilon}_{\mathsf{m}}, \tag{12}$$

where $\mathbb{C}_{\mathsf{m},\mathsf{M}}$ is a strictly negative constant which does not depend on ε.

Conditions at the Block-Fracture Interface. First, we assume the continuity of the phase fluxes and the pressures at the interface Σ^{ε}_{T}. Namely, we have :

$$\begin{cases} K_{\mathsf{f}}\lambda_{\mathsf{f},w}(S^{\varepsilon}_{\mathsf{f}})\nabla p^{\varepsilon}_{\mathsf{f},w}\cdot\boldsymbol{\nu} = \varepsilon^2\, K_{\mathsf{m}}\lambda_{\mathsf{m},w}(S^{\varepsilon}_{\mathsf{m}})\nabla p^{\varepsilon}_{\mathsf{m},w}\cdot\boldsymbol{\nu} \quad \text{on } \Sigma^{\varepsilon}_{T}; \\ K_{\mathsf{f}}\lambda_{\mathsf{f},n}(S^{\varepsilon}_{\mathsf{f}})\nabla p^{\varepsilon}_{\mathsf{f},n}\cdot\boldsymbol{\nu} = \varepsilon^2\, K_{\mathsf{m}}\lambda_{\mathsf{m},n}(S^{\varepsilon}_{\mathsf{m}})\nabla p^{\varepsilon}_{\mathsf{m},n}\cdot\boldsymbol{\nu} \quad \text{on } \Sigma^{\varepsilon}_{T}; \\ p^{\varepsilon}_{\mathsf{f},w} = p^{\varepsilon}_{\mathsf{m},w} \quad \text{and} \quad p^{\varepsilon}_{\mathsf{f},n} = p^{\varepsilon}_{\mathsf{m},n} \quad \text{on } \Sigma^{\varepsilon}_{T}, \end{cases} \tag{13}$$

where $\boldsymbol{\nu}$ is the unit outer normal vector on $\Gamma^{\varepsilon}_{\mathsf{fm}}$.

Remark 1. *Notice that the continuity of the phase pressures on the interface Σ^{ε}_{T} leads to the continuity of the capillary pressure function, i.e., $P_{\mathsf{f},c}(S^{\varepsilon}_{\mathsf{f}}) = P_{\mathsf{m},c}(S^{\varepsilon}_{\mathsf{m}})$ on Σ^{ε}_{T}. This relation, in particular, implies the discontinuity of the saturation function on the interface Σ^{ε}_{T}.*

Now we turn to the interface conditions for the concentration function. We assume the continuity of the concentration and the corresponding fluxes. Namely, we have :

$$\begin{cases} \mathsf{c}^{\varepsilon}_{\mathsf{f}} = \mathsf{c}^{\varepsilon}_{\mathsf{m}} \quad \text{on } \Sigma^{\varepsilon}_{T}; \\ \mathcal{D}_{\mathsf{f}}(S^{\varepsilon}_{\mathsf{m}})\nabla \mathsf{c}^{\varepsilon}_{\mathsf{f}}\cdot\boldsymbol{\nu} = \varepsilon^2\, \mathcal{D}_{\mathsf{m}}(S^{\varepsilon}_{\mathsf{m}})\nabla \mathsf{c}^{\varepsilon}_{\mathsf{m}}\cdot\boldsymbol{\nu} \quad \text{on } \Sigma^{\varepsilon}_{T}, \end{cases} \tag{14}$$

where $\boldsymbol{\nu}$ is the unit outer normal vector on $\Gamma^{\varepsilon}_{\mathsf{fm}}$.

Remark 2. *Notice that relation (13)$_2$ for the flux of the wetting phase pressure along with the relations (14) imply that*

$$\left\{K_{\mathsf{f}}\,\mathsf{c}_{\mathsf{f}}^{\varepsilon}\,\lambda_{\mathsf{f},w}(S_{\mathsf{f}}^{\varepsilon})\nabla p_{\mathsf{f},w}^{\varepsilon}+\mathcal{D}_{\mathsf{f}}(S_{\mathsf{f}}^{\varepsilon})\nabla\mathsf{c}_{\mathsf{f}}^{\varepsilon}\right\}\cdot\boldsymbol{\nu}=$$
$$=\varepsilon^{2}\left\{K_{\mathsf{m}}\,\mathsf{c}_{\mathsf{m}}^{\varepsilon}\,\lambda_{\mathsf{m},w}(S_{\mathsf{m}}^{\varepsilon})\nabla p_{\mathsf{m},w}^{\varepsilon}+\mathcal{D}_{\mathsf{m}}(S_{\mathsf{m}}^{\varepsilon})\nabla\mathsf{c}_{\mathsf{m}}^{\varepsilon}\right\}\cdot\boldsymbol{\nu}\quad\text{on }\ \Sigma_{T}^{\varepsilon}. \tag{15}$$

Initial Conditions. The initial conditions for the saturation function read :

$$S_{\mathsf{f}}^{\varepsilon}(x,0)=S_{\mathsf{f}}^{\mathrm{init}}(x)\quad\text{and}\quad S_{\mathsf{m}}^{\varepsilon}(x,0)=S_{\mathsf{m}}^{\mathrm{init}}(x). \tag{16}$$

The corresponding initial conditions for the concentration function have the form :

$$\mathsf{c}_{\mathsf{f}}^{\varepsilon}=\mathsf{c}_{\mathsf{f}}^{\mathrm{init}}(x)\quad\text{and}\quad\mathsf{c}_{\mathsf{m}}^{\varepsilon}=\mathsf{c}_{\mathsf{m}}^{\mathrm{init}}(x). \tag{17}$$

Finally, we notice that our equations along with the interface and initial conditions have to be completed by the boundary conditions on the external boundary $\partial\Omega$. However, since these conditions play no role in the homogenization process presented in this paper, then they will be omitted for the sake of brevity.

3 Upscaling of the two-phase flow model (5)- (17)

The outline of the section is as follows. First, in Section 3.1 we present briefly the well known method of two scale asymptotic expansions see, e.g., [5, 8, 9, 16, 19, 24]. In Section 3.2 we analyze the equations (5)$_1$ and (5)$_2$ in the fracture part. Section 3.3 is devoted to the asymptotic analysis of the interface conditions (13). In Section 3.4 we are dealing with the asymptotic analysis of equations (9)$_1$, (9)$_2$ in the matrix blocks. Section 3.5 is devoted to the asymptotic analysis of equation (5)$_3$ in fracture part. Then in Section 3.6 we carry out the asymptotic analysis of the interface conditions (15). Section 3.7 is devoted to the asymptotic analysis of equations (9)$_3$ in the matrix blocks. In Section 3.8 we establish the chain of equations, in particular, the homogenized equation (the equation coming from the zero order term in the formal asymptotic expansion). In Section 3.9, eliminating the corrector functions we obtain the homogenized equations involving only the zero order terms of the formal asymptotic expansions.

3.1 Two scale asymptotic expansions formalism

In this section, we use the method of two-length-scale asymptotic expansions, to derive the limiting equations in the case of a periodic structure. We will derive the macroscopic model of the flow in a formal sense. In what follows, we assume that the functions involved in (5)-(17) depend on the space variable in the following sense : **(i)** they are the functions of the macroscopic variable $x\in\Omega$ and **(ii)** for each $x\in\Omega$, of a microscopic variable $y\in Y\overset{\text{def}}{=}Y_{\mathsf{m}}\cup Y_{\mathsf{f}}\cup\Gamma_{\mathsf{fm}}$. The macroscopic and microscopic scales are related by the small parameter ε, i.e., up to a translation $y\overset{\text{def}}{=}\varepsilon^{-1}\,x$. This implies that $\nabla:=\nabla_x+\varepsilon^{-1}\,\nabla_y$, where ∇_x,∇_y are the gradients with respect to the variables x,y. Then the function u^{ε} standing for the phase pressures, saturation, concentration functions, and the density of the solid reagent is assumed to have the following asymptotic expansion (*ansatz*) :

$$u^{\varepsilon}(x,t)=u^{(0)}(x,y,t)+\varepsilon\,u^{(1)}(x,y,t)+\varepsilon^{2}\,u^{(2)}(x,y,t)+...=\sum_{j=0}^{+\infty}\varepsilon^{j}\,u^{(k)}(x,y,t), \tag{18}$$

where the function $u^{(k)}(x,y,t)$ is Y-periodic in y. Furthermore, the composition of functions $U(u^\varepsilon)$ can be expanded by using the Taylor theorem as follows :

$$U(u^\varepsilon) = U\big(u^{(0)}\big) + U'\big(u^{(0)}\big)\big(u^\varepsilon - u^{(0)}\big) + ... \stackrel{\text{def}}{=} U\big(u^{(0)}\big) + \varepsilon\, U^{(1)} + \varepsilon^2\, U^{(2)} + ...$$

From this equation, keeping the terms up to ε^2-order, we obtain the following formula :

$$U(u^\varepsilon) = U\big(u^{(0)}\big) + \varepsilon\, U'\big(u^{(0)}\big)u^{(1)} + \varepsilon^2\, \left[U'\big(u^{(0)}\big)u^{(2)} + \frac{1}{2}\, U''\big(u^{(0)}\big)u^{(1)}\right] + ... \tag{19}$$

Now taking into account the formalism (18), we introduce the two-scale asymptotic expansion of the functions $p^\varepsilon_{\ell,w}, p^\varepsilon_{\ell,n}, S^\varepsilon_\ell, \mathsf{c}^\varepsilon_\ell$, and $\mathsf{M}^\varepsilon_\ell$ ($\ell = \mathsf{f}, \mathsf{m}$) the solution to (5)-(17) :

$$p^\varepsilon_{\mathsf{f},\upsilon} = p^{(0)}_{\mathsf{f},\upsilon}(x,t) + \varepsilon\, p^{(1)}_{\mathsf{f},\upsilon}(x,y,t) + \varepsilon^2\, p^{(2)}_{\mathsf{f},\upsilon}(x,y,t) + ... \quad (\upsilon = w,n); \tag{20}$$

$$p^\varepsilon_{\mathsf{m},\upsilon} = p^{(0)}_{\mathsf{m},\upsilon}(x,y,t) + \varepsilon\, p^{(1)}_{\mathsf{m},\upsilon}(x,y,t) + \varepsilon^2\, p^{(2)}_{\mathsf{m},\upsilon}(x,y,t) + ... \quad (\upsilon = w,n); \tag{21}$$

$$S^\varepsilon_{\mathsf{f}} = S^{(0)}_{\mathsf{f}}(x,t) + \varepsilon\, S^{(1)}_{\mathsf{f}}(x,y,t) + \varepsilon^2\, S^{(2)}_{\mathsf{f}}(x,y,t) + ... \ ; \tag{22}$$

$$S^\varepsilon_{\mathsf{m}} = S^{(0)}_{\mathsf{m}}(x,y,t) + \varepsilon\, S^{(1)}_{\mathsf{m}}(x,y,t) + \varepsilon^2\, S^{(2)}_{\mathsf{m}}(x,y,t) + ... \ ; \tag{23}$$

$$\mathsf{c}^\varepsilon_{\mathsf{f}} = \mathsf{c}^{(0)}_{\mathsf{f}}(x,t) + \varepsilon\, \mathsf{c}^{(1)}_{\mathsf{f}}(x,y,t) + \varepsilon^2\, \mathsf{c}^{(2)}_{\mathsf{f}}(x,y,t) + ... \ ; \tag{24}$$

$$\mathsf{c}^\varepsilon_{\mathsf{m}} = \mathsf{c}^{(0)}_{\mathsf{m}}(x,y,t) + \varepsilon\, \mathsf{c}^{(1)}_{\mathsf{m}}(x,y,t) + \varepsilon^2\, \mathsf{c}^{(2)}_{\mathsf{m}}(x,y,t) + ... \ ; \tag{25}$$

$$\mathsf{M}^\varepsilon_{\mathsf{f}} = \mathsf{M}^{(0)}_{\mathsf{f}}(x,t) + \varepsilon\, \mathsf{M}^{(1)}_{\mathsf{f}}(x,y,t) + \varepsilon^2\, \mathsf{M}^{(2)}_{\mathsf{f}}(x,y,t) + ... \ ; \tag{26}$$

$$\mathsf{M}^\varepsilon_{\mathsf{m}} = \mathsf{M}^{(0)}_{\mathsf{m}}(x,y,t) + \varepsilon\, \mathsf{M}^{(1)}_{\mathsf{m}}(x,y,t) + \varepsilon^2\, \mathsf{M}^{(2)}_{\mathsf{m}}(x,y,t) + ... \tag{27}$$

Remark 3. *Note that the first terms $p^{(0)}_{\mathsf{f},\upsilon}$ $(\upsilon = w,n)$, $S^{(0)}_{\mathsf{f}}, \mathsf{c}^{(0)}_{\mathsf{f}}$, and $\mathsf{M}^{(0)}_{\mathsf{f}}$ do not depend on the fast variable y, which is explained by the fact that the saturation field in fractures is instantaneously established and, consequently, is uniform within a cell (see, for instance, [5, 21]).*

Now we turn to the asymptotical expansions of the mobility functions. The functions $\lambda_{\ell,w}, \lambda_{\ell,n}$ depend in a nonlinear way on the saturation S^ε_ℓ. In order to obtain the asymptotic expansions for these functions, we make use of the formula (19). We have :

$$\lambda_{\ell,\upsilon}(S^\varepsilon_\ell) = \lambda_{\ell,\upsilon}\left(S^{(0)}_\ell\right) + \varepsilon\, \lambda'_{\ell,\upsilon}\left(S^{(0)}_\ell\right)\, S^{(1)}_\ell + \varepsilon^2\, \mathsf{L}^{(2)}_{\ell,\upsilon} + ... \quad (\ell = \mathsf{f},\mathsf{m};\ \upsilon = w,n), \tag{28}$$

where

$$\mathsf{L}^{(2)}_{\ell,\upsilon} \stackrel{\text{def}}{=} \lambda'_{\ell,\upsilon}\left(S^{(0)}_\ell\right) S^{(2)}_\ell + \frac{1}{2}\, \lambda''_{\ell,w}\left(S^{(0)}_\ell\right) S^{(1)}_\ell \quad (\ell = \mathsf{f},\mathsf{m};\ \upsilon = w,n). \tag{29}$$

In a similar way we can obtain the asymptotical expansion of the diffusion coefficient $\mathcal{D}^\varepsilon_\ell = \mathcal{D}_\ell(S^\varepsilon_{\ell,w})$. Namely, we have :

$$\mathcal{D}_\ell(S^\varepsilon_\ell) = \mathcal{D}_\ell\left(S^{(0)}_\ell\right) + \varepsilon\, \mathcal{D}'_\ell\left(S^{(0)}_\ell\right)\, S^{(1)}_\ell + \varepsilon^2\, \mathcal{D}^{(2)}_\ell + ... \quad (\ell = \mathsf{f},\mathsf{m}), \tag{30}$$

where

$$\mathcal{D}^{(2)}_\ell \stackrel{\text{def}}{=} \mathcal{D}'_\ell\left(S^{(0)}_\ell\right)\, S^{(2)}_\ell + \frac{1}{2}\, \mathcal{D}''_\ell\left(S^{(0)}_\ell\right)\, S^{(1)}_\ell \quad (\ell = \mathsf{f},\mathsf{m}). \tag{31}$$

3.2 Asymptotic analysis of equations $(5)_1$ and $(5)_2$ in the fracture part

We start our analysis by considering equation $(5)_1$. Plugging (20), (22), (24) in $(5)_1$, using (28) along with the differential law $\nabla := \nabla_x + \varepsilon^{-1}\nabla_y$, we get :

$$\Phi_{\mathsf{f}}\frac{\partial}{\partial t}\left(S_{\mathsf{f}}^{(0)}(x,t)+\varepsilon\, S_{\mathsf{f}}^{(1)}(x,y,t)+....\right)-$$

$$-K_{\mathsf{f}}\left\{\mathrm{div}_x+\frac{1}{\varepsilon}\mathrm{div}_y\right\}\left\{\left[\lambda_{\mathsf{f},w}\left(S_{\mathsf{f}}^{(0)}\right)+\varepsilon\lambda'_{\mathsf{f},w}\left(S_{\mathsf{f}}^{(0)}\right)S_{\mathsf{f}}^{(1)}+...\right]\left[\left\{\nabla_x+\frac{1}{\varepsilon}\nabla_y\right\}\left(p_{\mathsf{f},w}^{(0)}+\varepsilon p_{\mathsf{f},w}^{(1)}+...\right)\right]\right\}=$$

$$=\mathbb{C}_{\mathsf{f},w}\left(S_{\mathsf{f}}^{(0)}(x,t)+...\right)\left(\mathsf{M}_{\mathsf{f}}^{(0)}(x,t)+...\right)\left(\mathsf{c}_{\mathsf{f}}^{(0)}(x,t)+...\right). \tag{32}$$

The divergence operator in (32) we rewrite as follows :

$$\left\{\mathrm{div}_x+\frac{1}{\varepsilon}\,\mathrm{div}_y\right\}\left[\left(\lambda_{\mathsf{f},w}\left(S_{\mathsf{f}}^{(0)}\right)+...\right)\left\{\nabla_x+\frac{1}{\varepsilon}\nabla_y\right\}\left(p_{\mathsf{f},w}^{(0)}+...\right)\right]=$$

$$=\mathrm{div}_x\left\{\left(\lambda_{\mathsf{f},w}\left(S_{\mathsf{f}}^{(0)}\right)+...\right)\nabla_x\left(p_{\mathsf{f},w}^{(0)}+...\right)\right\}+\frac{1}{\varepsilon}\,\mathrm{div}_x\left\{\left(\lambda_{\mathsf{f},w}\left(S_{\mathsf{f}}^{(0)}\right)+...\right)\nabla_y\left(p_{\mathsf{f},w}^{(0)}+...\right)\right\}+$$

$$+\frac{1}{\varepsilon}\,\mathrm{div}_y\left\{\left(\lambda_{\mathsf{f},w}\left(S_{\mathsf{f}}^{(0)}\right)+...\right)\nabla_x\left(p_{\mathsf{f},w}^{(0)}+...\right)\right\}+\frac{1}{\varepsilon^2}\,\mathrm{div}_y\left\{\left(\lambda_{\mathsf{f},w}\left(S_{\mathsf{f}}^{(0)}\right)+...\right)\nabla_y\left(p_{\mathsf{f},w}^{(0)}+...\right)\right\}. \tag{33}$$

Now let us represent equation (32) as a power series with respect to the parameter ε. We have :

$$\boldsymbol{\varepsilon^{-2}}\left[-K_{\mathsf{f}}\,\mathrm{div}_y\left\{\lambda_{\mathsf{f},w}\left(S_{\mathsf{f}}^{(0)}\right)\nabla_y p_{\mathsf{f},w}^{(0)}\right\}\right]+$$

$$+\boldsymbol{\varepsilon^{-1}}\left[-K_{\mathsf{f}}\,\mathrm{div}_x\left\{\lambda_{\mathsf{f},w}\left(S_{\mathsf{f}}^{(0)}\right)\nabla_y p_{\mathsf{f},w}^{(0)}\right\}-K_{\mathsf{f}}\,\mathrm{div}_y\left\{\lambda_{\mathsf{f},w}\left(S_{\mathsf{f}}^{(0)}\right)\nabla_x p_{\mathsf{f},w}^{(0)}\right\}-\right.$$

$$\left.-K_{\mathsf{f}}\,\mathrm{div}_y\left\{\lambda'_{\mathsf{f},w}\left(S_{\mathsf{f}}^{(0)}\right)S_{\mathsf{f}}^{(1)}\nabla_y p_{\mathsf{f},w}^{(0)}+\lambda_{\mathsf{f},w}\left(S_{\mathsf{f}}^{(0)}\right)\nabla_y p_{\mathsf{f},w}^{(1)}\right\}\right]+$$

$$+\boldsymbol{\varepsilon^{0}}\left[\Phi_{\mathsf{f}}\frac{\partial S_{\mathsf{f}}^{(0)}}{\partial t}-K_{\mathsf{f}}\,\mathrm{div}_x\left\{\lambda_{\mathsf{f},w}\left(S_{\mathsf{f}}^{(0)}\right)\nabla_x p_{\mathsf{f},w}^{(0)}\right\}-K_{\mathsf{f}}\,\mathrm{div}_x\left\{\lambda'_{\mathsf{f},w}\left(S_{\mathsf{f}}^{(0)}\right)S_{\mathsf{f}}^{(1)}\nabla_y p_{\mathsf{f},w}^{(0)}\right\}-\right.$$

$$-K_{\mathsf{f}}\,\mathrm{div}_x\left\{\lambda_{\mathsf{f},w}\left(S_{\mathsf{f}}^{(0)}\right)\nabla_y p_{\mathsf{f},w}^{(1)}\right\}-K_{\mathsf{f}}\,\mathrm{div}_y\left\{\lambda'_{\mathsf{f},w}\left(S_{\mathsf{f}}^{(0)}\right)S_{\mathsf{f}}^{(1)}\nabla_x p_{\mathsf{f},w}^{(0)}\right\}-K_{\mathsf{f}}\,\mathrm{div}_y\left\{\lambda_{\mathsf{f},w}\left(S_{\mathsf{f}}^{(0)}\right)\nabla_x p_{\mathsf{f},w}^{(1)}\right\}-$$

$$\left.-\frac{1}{2}K_{\mathsf{f}}\mathrm{div}_y\left\{\mathsf{L}_{\mathsf{f},w}^{(2)}\nabla_y p_{\mathsf{f},w}^{(0)}\right\}-K_{\mathsf{f}}\mathrm{div}_y\left\{\lambda_{\mathsf{f},w}\left(S_{\mathsf{f}}^{(0)}\right)\nabla_y p_{\mathsf{f},w}^{(2)}\right\}-K_{\mathsf{f}}\mathrm{div}_y\left\{\lambda'_{\mathsf{f},w}\left(S_{\mathsf{f}}^{(0)}\right)S_{\mathsf{f}}^{(1)}\nabla_y p_{\mathsf{f},w}^{(1)}\right\}\right]+....=$$

$$=\boldsymbol{\varepsilon^{0}}\left[\mathbb{C}_{\mathsf{f},w}\,S_{\mathsf{f}}^{(0)}(x,t)\,\mathsf{M}_{\mathsf{f}}^{(0)}(x,t)\mathsf{c}_{\mathsf{f}}^{(0)}(x,t)\right]+....$$

Now taking into account that $p_{\mathsf{f},w}^{(0)}, S_{\mathsf{f}}^{(0)}$ do not depend on the fast variable y, we get :

$$\boldsymbol{\varepsilon^{-1}}\left[-K_{\mathsf{f}}\,\mathrm{div}_y\left\{\lambda_{\mathsf{f},w}\left(S_{\mathsf{f}}^{(0)}\right)\nabla_x p_{\mathsf{f},w}^{(0)}\right\}-K_{\mathsf{f}}\,\mathrm{div}_y\left\{\lambda_{\mathsf{f},w}\left(S_{\mathsf{f}}^{(0)}\right)\nabla_y p_{\mathsf{f},w}^{(1)}\right\}\right]+$$

$$+\boldsymbol{\varepsilon^{0}}\left[\Phi_{\mathsf{f}}\frac{\partial S_{\mathsf{f}}^{(0)}}{\partial t}-K_{\mathsf{f}}\,\mathrm{div}_x\left\{\lambda_{\mathsf{f},w}\left(S_{\mathsf{f}}^{(0)}\right)\left(\nabla_x p_{\mathsf{f},w}^{(0)}+\nabla_y p_{\mathsf{f},w}^{(1)}\right)\right\}\right.$$

$$-K_{\mathsf{f}}\,\mathrm{div}_y\Big\{\lambda'_{\mathsf{f},w}\left(S_{\mathsf{f}}^{(0)}\right)S_{\mathsf{f}}^{(1)}\left(\nabla_x p_{\mathsf{f},w}^{(0)}+\nabla_y p_{\mathsf{f},w}^{(1)}\right)+\lambda_{\mathsf{f},w}\left(S_{\mathsf{f}}^{(0)}\right)\left(\nabla_x p_{\mathsf{f},w}^{(1)}+\nabla_y p_{\mathsf{f},w}^{(2)}\right)\Big\}\Big]+...=$$

$$=\boldsymbol{\varepsilon^0}\left[\mathbb{C}_{\mathsf{f},w}\,S_{\mathsf{f}}^{(0)}(x,t)\,\mathsf{M}_{\mathsf{f}}^{(0)}(x,t)\mathsf{c}_{\mathsf{f}}^{(0)}(x,t)\right]+.... \tag{34}$$

The asymptotic analysis of equation $(5)_2$ can be done in a similar way.

3.3 Asymptotic analysis of the interface conditions (13)

We start by considering the interface condition $(13)_1$. We plug the asymptotic expansions (20), (22) in $(13)_1$ to have :

$$K_{\mathsf{f}}\left[\lambda_{\mathsf{f},w}\Big(S_{\mathsf{f}}^{(0)}\Big)+\varepsilon\,\lambda'_{\mathsf{f},w}\Big(S_{\mathsf{f}}^{(0)}\Big)\,S_{\mathsf{f}}^{(1)}...\right]\left\{\left(\nabla_x+\frac{1}{\varepsilon}\nabla_y\right)\left(p_{\mathsf{f},w}^{(0)}+\varepsilon\,p_{\mathsf{f},w}^{(1)}+...\right)\right\}\cdot\boldsymbol{\nu}=$$

$$=\varepsilon^2\,K_{\mathsf{m}}\left[\lambda_{\mathsf{m},w}\left(S_{\mathsf{m}}^{(0)}\right)+\varepsilon...\right]\left\{\left(\nabla_x+\frac{1}{\varepsilon}\nabla_y\right)\left(p_{\mathsf{m},w}^{(0)}+\varepsilon\,p_{\mathsf{m},w}^{(1)}+...\right)\right\}\cdot\boldsymbol{\nu}. \tag{35}$$

Now we can represent the relation (35) as a series with respect to the parameter ε. We have :

$$\boldsymbol{\varepsilon^{-1}}\left[K_{\mathsf{f}}\,\lambda_{\mathsf{f},w}\left(S_{\mathsf{f}}^{(0)}\right)\nabla_y p_{\mathsf{f},w}^{(0)}\cdot\boldsymbol{\nu}\right]+\boldsymbol{\varepsilon^0}\left[K_{\mathsf{f}}\,\lambda_{\mathsf{f},w}\Big(S_{\mathsf{f}}^{(0)}\Big)\Big(\nabla_x p_{\mathsf{f},w}^{(0)}+\nabla_y p_{\mathsf{f},w}^{(1)}\Big)\cdot\boldsymbol{\nu}\right]+$$

$$+\boldsymbol{\varepsilon^1}\left[K_{\mathsf{f}}\left(\lambda_{\mathsf{f},w}\Big(S_{\mathsf{f}}^{(0)}\Big)\nabla_x p_{\mathsf{f},w}^{(1)}+\lambda_{\mathsf{f},w}\Big(S_{\mathsf{f}}^{(0)}\Big)\nabla_y p_{\mathsf{f},w}^{(2)}+\lambda'_{\mathsf{f},w}\Big(S_{\mathsf{f}}^{(0)}\Big)\,S_{\mathsf{f}}^{(1)}\nabla_x p_{\mathsf{f},w}^{(0)}+\lambda'_{\mathsf{f},w}\Big(S_{\mathsf{f}}^{(0)}\Big)\,S_{\mathsf{f}}^{(1)}\nabla_y p_{\mathsf{f},w}^{(1)}\right)\cdot\boldsymbol{\nu}-$$

$$-K_{\mathsf{m}}\,\lambda_{\mathsf{m},w}\left(S_{\mathsf{m}}^{(0)}\right)\nabla_y p_{\mathsf{m},w}^{(0)}\cdot\boldsymbol{\nu}\Big]+...=0.$$

Now taking into account that $p_{\mathsf{f},w}^{(0)}, S_{\mathsf{f}}^{(0)}$ do not depend on y, from this relation we get :

$$\boldsymbol{\varepsilon^0}\left[K_{\mathsf{f}}\,\lambda_{\mathsf{f},w}\Big(S_{\mathsf{f}}^{(0)}\Big)\Big(\nabla_x p_{\mathsf{f},w}^{(0)}+\nabla_y p_{\mathsf{f},w}^{(1)}\Big)\cdot\boldsymbol{\nu}\right]+$$

$$+\boldsymbol{\varepsilon^1}\left[K_{\mathsf{f}}\lambda_{\mathsf{f},w}\Big(S_{\mathsf{f}}^{(0)}\Big)\Big(\nabla_x p_{\mathsf{f},w}^{(1)}+\nabla_y p_{\mathsf{f},w}^{(2)}\Big)\cdot\boldsymbol{\nu}+K_{\mathsf{f}}\lambda'_{\mathsf{f},w}\Big(S_{\mathsf{f}}^{(0)}\Big)\Big(\nabla_x p_{\mathsf{f},w}^{(0)}+\nabla_y p_{\mathsf{f},w}^{(1)}\Big)\,S_{\mathsf{f}}^{(1)}\cdot\boldsymbol{\nu}-\right.$$

$$\left.-K_{\mathsf{m}}\,\lambda_{\mathsf{m},w}\left(S_{\mathsf{m}}^{(0)}\right)\nabla_y p_{\mathsf{m},w}^{(0)}\cdot\boldsymbol{\nu}\right]+...=0. \tag{36}$$

The asymptotic analysis of equation $(13)_2$ can be done in a similar way.

Finally, we turn to the interface condition $(13)_3$. Plugging the asymptotic expansions (20), (22) in $(13)_3$ one easily gets the following asymptotic expansions :

$$\boldsymbol{\varepsilon^0}\left[p_{\mathsf{f},w}^{(0)}-p_{\mathsf{m},w}^{(0)}\right]+...=0\quad\text{and}\quad\boldsymbol{\varepsilon^0}\left[p_{\mathsf{f},n}^{(0)}-p_{\mathsf{m},n}^{(0)}\right]+...=0. \tag{37}$$

3.4 Asymptotic analysis of equations $(9)_1$, $(9)_2$ in the matrix blocks

Plugging the asymptotic expansions (21), (23) in $(9)_1$, $(9)_2$ and restricting ourselves to the terms of order ε^0, we easily get :

$$\boldsymbol{\varepsilon^0}\left[\Phi_{\mathsf{m}}\frac{\partial S^{(0)}_{\mathsf{m}}}{\partial t}-K_{\mathsf{m}}\,\mathrm{div}_y\left\{\lambda_{\mathsf{m},w}(S^{(0)}_{\mathsf{m}})\nabla p^{(0)}_{\mathsf{m},w}\right\}\right]+...\;=\;\boldsymbol{\varepsilon^0}\,\mathbb{S}^{(0)}_{\mathsf{m},w}+... \tag{38}$$

and

$$\boldsymbol{\varepsilon^0}\left[\Phi_{\mathsf{m}}\frac{\partial S^{(0)}_{\mathsf{m}}}{\partial t}-K_{\mathsf{m}}\,\mathrm{div}_y\left\{\lambda_{\mathsf{m},n}(S^{(0)}_{\mathsf{m}})\nabla p^{(0)}_{\mathsf{m},n}\right\}\right]+...\;=\;\boldsymbol{\varepsilon^0}\,\mathbb{S}^{(0)}_{\mathsf{m},n}+..., \tag{39}$$

where by (11), (23), (25), (27), the source terms $\mathbb{S}^{(0)}_{\mathsf{m},w}, \mathbb{S}^{(0)}_{\mathsf{m},n}$ are given by :

$$\mathbb{S}^{(0)}_{\mathsf{m},w}\overset{\text{def}}{=}\mathbb{C}_{\mathsf{m},w}\,S^{(0)}_{\mathsf{m}}\,\mathsf{M}^{(0)}_{\mathsf{m}}\,\mathsf{c}^{(0)}_{\mathsf{m}}\quad\text{and}\quad\mathbb{S}^{(0)}_{\mathsf{m},n}\overset{\text{def}}{=}\mathbb{C}_{\mathsf{m},n}\left[1-S^{(0)}_{\mathsf{m}}\right]\mathsf{M}^{(0)}_{\mathsf{m}}\,\mathsf{c}^{(0)}_{\mathsf{m}}. \tag{40}$$

3.5 Asymptotic analysis of equation $(5)_3$ in fracture part

We recall that the equation $(5)_3$ reads :

$$\Phi^{\varepsilon}(x)\frac{\partial}{\partial t}\left[S^{\varepsilon}_{\mathsf{f}}\,\mathsf{c}^{\varepsilon}_{\mathsf{f}}\right]-\Upsilon^{\varepsilon}_1-\Upsilon^{\varepsilon}_2=\mathbb{S}^{\varepsilon}_{\mathsf{f},w}, \tag{41}$$

where

$$\Upsilon^{\varepsilon}_1\overset{\text{def}}{=}\mathrm{div}\left\{K_{\mathsf{f}}\,\mathsf{c}^{\varepsilon}_{\mathsf{f}}\,\lambda_{\mathsf{f},w}(S^{\varepsilon}_{\mathsf{f}})\nabla p^{\varepsilon}_{\mathsf{f},w}\right\}\quad\text{and}\quad\Upsilon^{\varepsilon}_2\overset{\text{def}}{=}\mathrm{div}\left\{\mathcal{D}_{\mathsf{f}}(S^{\varepsilon}_{\mathsf{f}})\nabla\mathsf{c}^{\varepsilon}_{\mathsf{f}}\right\}. \tag{42}$$

Asymptotic behavior of Υ^{ε}_1. Let us introduce the notation :

$$\mathcal{P}^{\varepsilon}\overset{\text{def}}{=}K_{\mathsf{f}}\,\lambda_{\mathsf{f},w}(S^{\varepsilon}_{\mathsf{f}})\nabla p^{\varepsilon}_{\mathsf{f},w}. \tag{43}$$

Then from (28)-(29) and (20), (22), we have that

$$\mathcal{P}^{\varepsilon}=K_{\mathsf{f}}\left[\lambda_{\mathsf{f},w}\left(S^{(0)}_{\mathsf{f}}\right)+\varepsilon\,\lambda'_{\mathsf{f},w}\left(S^{(0)}_{\mathsf{f}}\right)\,S^{(1)}_{\mathsf{f}}+\varepsilon^2\,\mathsf{L}^{(2)}_{\mathsf{f},w}+...\right]\times$$

$$\times\left[\varepsilon^{-1}\left(\nabla_y p^{(0)}_{\mathsf{f},w}\right)+\varepsilon^0\left(\nabla_x p^{(0)}_{\mathsf{f},w}+\nabla_y p^{(1)}_{\mathsf{f},w}\right)+\varepsilon^1\left(\nabla_x p^{(1)}_{\mathsf{f},w}+\nabla_y p^{(2)}_{\mathsf{f},w}\right)+...\right].$$

Now taking into account that the function $p^{(0)}_{\mathsf{f},w}, S^{(0)}_{\mathsf{f}}$ do not depend on the fast variable y, from this relation (up to the terms of order ε^1) we get :

$$\mathcal{P}^{\varepsilon}=\boldsymbol{\varepsilon^0}\,\left[K_{\mathsf{f}}\lambda_{\mathsf{f},w}\left(S^{(0)}_{\mathsf{f}}\right)\left(\nabla_x p^{(0)}_{\mathsf{f},w}+\nabla_y p^{(1)}_{\mathsf{f},w}\right)\right]+$$

$$+\boldsymbol{\varepsilon^1}\,\left[K_{\mathsf{f}}\lambda_{\mathsf{f},w}\left(S^{(0)}_{\mathsf{f}}\right)\left(\nabla_x p^{(1)}_{\mathsf{f},w}+\nabla_y p^{(2)}_{\mathsf{f},w}\right)+K_{\mathsf{f}}\,S^{(1)}_{\mathsf{f}}\,\lambda'_{\mathsf{f},w}\left(S^{(0)}_{\mathsf{f}}\right)\,\left(\nabla_x p^{(0)}_{\mathsf{f},w}+\nabla_y p^{(1)}_{\mathsf{f},w}\right)\right]+...\overset{\text{def}}{=}$$

$$\overset{\text{def}}{=}\boldsymbol{\varepsilon^0}\,\mathcal{P}^{(0)}+\boldsymbol{\varepsilon^1}\,\mathcal{P}^{(1)}+... \tag{44}$$

Now, from (24), (41), (44) we obtain that

$$\mathcal{P}^{\varepsilon}\,\mathsf{c}^{\varepsilon}_{\mathsf{f}}=\left(\varepsilon^0\,\mathcal{P}^{(0)}+\varepsilon^1\,\mathcal{P}^{(1)}+...\right)\,\left(\varepsilon^0\,\mathsf{c}^{(0)}_{\mathsf{f}}+\varepsilon\,\mathsf{c}^{(1)}_{\mathsf{f}}+...\right)=$$

$$= \boldsymbol{\varepsilon^0}\ \left[\mathsf{c}_\mathsf{f}^{(0)} K_\mathsf{f} \lambda_{\mathsf{f},w}\left(S_\mathsf{f}^{(0)}\right)\left(\nabla_x p_{\mathsf{f},w}^{(0)} + \nabla_y p_{\mathsf{f},w}^{(1)}\right)\right] +$$

$$+\boldsymbol{\varepsilon^1}\ \left\{\mathsf{c}_\mathsf{f}^{(0)}\left[K_\mathsf{f}\lambda_{\mathsf{f},w}\left(S_\mathsf{f}^{(0)}\right)\left(\nabla_x p_{\mathsf{f},w}^{(1)} + \nabla_y p_{\mathsf{f},w}^{(2)}\right) + K_\mathsf{f}\lambda'_{\mathsf{f},w}\left(S_\mathsf{f}^{(0)}\right)\ S_\mathsf{f}^{(1)}\left(\nabla_x p_{\mathsf{f},w}^{(0)} + \nabla_y p_{\mathsf{f},w}^{(1)}\right)\right] +\right.$$

$$\left. +\mathsf{c}_\mathsf{f}^{(1)}\left[K_\mathsf{f}\lambda_{\mathsf{f},w}\left(S_\mathsf{f}^{(0)}\right)\left(\nabla_x p_{\mathsf{f},w}^{(0)} + \nabla_y p_{\mathsf{f},w}^{(1)}\right)\right]\right\} + ... \stackrel{\text{def}}{=} \boldsymbol{\varepsilon^0}\,\mathcal{P}_{\mathsf{c}_\mathsf{f}}^{(0)} + \boldsymbol{\varepsilon^1}\,\mathcal{P}_{\mathsf{c}_\mathsf{f}}^{(1)} + ... \tag{45}$$

The relation (45) enables to obtain the asymptotic expansion of Υ_1^ε. Namely, we have :

$$\Upsilon_1^\varepsilon = \left\{\mathrm{div}_x + \frac{1}{\varepsilon}\,\mathrm{div}_y\right\}\left(K_\mathsf{f}\,\mathsf{c}_\mathsf{f}^\varepsilon\,\lambda_{\mathsf{f},w}(S_\mathsf{f}^\varepsilon)\nabla p_{\mathsf{f},w}^\varepsilon\right) = \boldsymbol{\varepsilon^0}\,\mathrm{div}_x\mathcal{P}_{\mathsf{c}_\mathsf{f}}^{(0)} + \boldsymbol{\varepsilon^0}\,\mathrm{div}_y\mathcal{P}_{\mathsf{c}_\mathsf{f}}^{(1)} + ... \tag{46}$$

Asymptotic behavior of Υ_2^ε. Proceeding now as above and taking into account that the function $\mathsf{c}_\mathsf{f}^{(0)}$ does not depend on y, we get :

$$\Upsilon_2^\varepsilon = \boldsymbol{\varepsilon^{-1}}\ \left[\mathrm{div}_y\left\{\mathcal{D}_\mathsf{f}\left(S_\mathsf{f}^{(0)}\right)\nabla_x\mathsf{c}_\mathsf{f}^{(0)}\right\} + \mathrm{div}_y\left\{\mathcal{D}_\mathsf{f}\left(S_\mathsf{f}^{(0)}\right)\nabla_y\mathsf{c}_\mathsf{f}^{(1)}\right\}\right] +$$

$$+\boldsymbol{\varepsilon^0}\,\mathrm{div}_x\left\{\mathcal{D}_\mathsf{f}\left(S_\mathsf{f}^{(0)}\right)\left(\nabla_x\mathsf{c}_\mathsf{f}^{(0)} + \nabla_y\mathsf{c}_\mathsf{f}^{(1)}\right)\right\}+$$

$$+\boldsymbol{\varepsilon^0}\,\mathrm{div}_y\left\{\mathcal{D}'_\mathsf{f}\left(S_\mathsf{f}^{(0)}\right)S_\mathsf{f}^{(1)}\ \left(\nabla_x\mathsf{c}_\mathsf{f}^{(0)} + \nabla_y\mathsf{c}_\mathsf{f}^{(1)}\right) + \mathcal{D}_\mathsf{f}\left(S_\mathsf{f}^{(0)}\right)\left(\nabla_x\mathsf{c}_\mathsf{f}^{(1)} + \nabla_y\mathsf{c}_\mathsf{f}^{(2)}\right)\right\} + ... \tag{47}$$

Let us introduce the notation :

$$\mathcal{C}^{(0)} \stackrel{\text{def}}{=} \mathcal{D}_\mathsf{f}\left(S_\mathsf{f}^{(0)}\right)\left(\nabla_x\mathsf{c}_\mathsf{f}^{(0)} + \nabla_y\mathsf{c}_\mathsf{f}^{(1)}\right); \tag{48}$$

$$\mathcal{C}^{(1)} \stackrel{\text{def}}{=} \mathcal{D}'_\mathsf{f}\left(S_\mathsf{f}^{(0)}\right)S_\mathsf{f}^{(1)}\ \left(\nabla_x\mathsf{c}_\mathsf{f}^{(0)} + \nabla_y\mathsf{c}_\mathsf{f}^{(1)}\right) + \mathcal{D}_\mathsf{f}\left(S_\mathsf{f}^{(0)}\right)\left(\nabla_x\mathsf{c}_\mathsf{f}^{(1)} + \nabla_y\mathsf{c}_\mathsf{f}^{(2)}\right). \tag{49}$$

Taking into account (48), (49), we rewrite (47) as follows :

$$\Upsilon_2^\varepsilon = \boldsymbol{\varepsilon^{-1}}\,\mathrm{div}_y\left\{\mathcal{D}_\mathsf{f}\left(S_\mathsf{f}^{(0)}\right)\left(\nabla_x\mathsf{c}_\mathsf{f}^{(0)} + \nabla_y\mathsf{c}_\mathsf{f}^{(1)}\right)\right\} + \boldsymbol{\varepsilon^0}\,\mathrm{div}_x\mathcal{C}^{(0)} + \boldsymbol{\varepsilon^0}\,\mathrm{div}_y\mathcal{C}^{(1)} + ... \tag{50}$$

Finally, from (46) and (50) we get :

$$\Upsilon_1^\varepsilon + \Upsilon_2^\varepsilon = \boldsymbol{\varepsilon^{-1}}\,\mathrm{div}_y\left\{\mathcal{D}_\mathsf{f}\left(S_\mathsf{f}^{(0)}\right)\left(\nabla_x\mathsf{c}_\mathsf{f}^{(0)} + \nabla_y\mathsf{c}_\mathsf{f}^{(1)}\right)\right\}+$$

$$+\boldsymbol{\varepsilon^0}\,\mathrm{div}_x\left\{\mathcal{C}^{(0)} + \mathcal{P}_{\mathsf{c}_\mathsf{f}}^{(0)}\right\} + \boldsymbol{\varepsilon^0}\,\mathrm{div}_y\left\{\mathcal{C}^{(1)} + \mathcal{P}_{\mathsf{c}_\mathsf{f}}^{(1)}\right\} + ... \tag{51}$$

3.6 Asymptotic analysis of the interface condition (15)

As in Section 3.3 we have that (see also the relation (51))

$$\boldsymbol{\varepsilon^1}\left\{\left[\mathcal{C}^{(1)} + \mathcal{P}_{\mathsf{c}_\mathsf{f}}^{(1)}\right] - \left[K_\mathsf{m}\,\mathsf{c}_\mathsf{m}^{(0)}\lambda_{\mathsf{m},w}\left(S_\mathsf{m}^{(0)}\right)\nabla_y p_{\mathsf{m},w}^{(0)} + \mathcal{D}_\mathsf{m}\left(S_\mathsf{m}^{(0)}\right)\nabla_y\mathsf{c}_\mathsf{m}^{(0)}\right]\right\}\cdot\boldsymbol{\nu} + ... = 0. \tag{52}$$

Finally, we turn to the interface condition $(14)_1$. We get :

$$\boldsymbol{\varepsilon^0}\ \left[\mathsf{c}_\mathsf{f}^{(0)} - \mathsf{c}_\mathsf{m}^{(0)}\right] + ... = 0. \tag{53}$$

3.7 Asymptotic analysis of equations $(9)_3$ in the matrix blocks

Plugging the asymptotic expansions (21), (23), (25) in $(9)_3$ and restricting ourselves to the terms of order ε^0, we easily get :

$$\boldsymbol{\varepsilon^0}\ \left[\Phi_\mathsf{m}\,\frac{\partial}{\partial t}\left[S_\mathsf{m}^{(0)}\,\mathsf{c}_\mathsf{m}^{(0)}\right] - K_\mathsf{m}\,\mathrm{div}_y\left\{K_\mathsf{m}\,\mathsf{c}_\mathsf{m}^{(0)}\lambda_{\mathsf{m},w}\left(S_\mathsf{m}^{(0)}\right)\nabla_y p_{\mathsf{m},w}^{(0)} + \mathcal{D}_\mathsf{m}\left(S_\mathsf{m}^{(0)}\right)\nabla_y \mathsf{c}_\mathsf{m}^{(0)}\right\}\right] + ... =$$

$$= \ \boldsymbol{\varepsilon^0}\,\mathbb{S}_{\mathsf{m},w}^{(0)} + ..., \tag{54}$$

where $\mathbb{S}_{\mathsf{m},w}^{(0)}$ is defined in (40).

3.8 Chain of equations

In this section, using the results of the previous ones, we are led to the following relations :

$\boldsymbol{\varepsilon^{-2}}$–equations. Taking into account our standing assumptions on the zero terms of the asymptotic expansions, i.e., $p_{\mathsf{f},w}^{(0)} = p_{\mathsf{f},w}^{(0)}(x,t)$, $p_{\mathsf{f},n}^{(0)} = p_{\mathsf{f},n}^{(0)}(x,t)$, $S_\mathsf{f}^{(0)} = S_\mathsf{f}^{(0)}(x,t)$, and $\mathsf{c}_\mathsf{f}^{(0)} = \mathsf{c}_\mathsf{f}^{(0)}(x,t)$, we observe that the $\boldsymbol{\varepsilon^{-2}}$–equations are satisfied automatically.

$\boldsymbol{\varepsilon^{-1}}$– equations. From the second term on the left-hand side of (34) and taking into account that $p_{\mathsf{f},w}^{(0)} = p_{\mathsf{f},w}^{(0)}(x,t)$, $S_\mathsf{f}^{(0)} = S_\mathsf{f}^{(0)}(x,t)$, we get the following equation for the corrector function $p_{\mathsf{f},w}^{(1)}$:

$$-\mathrm{div}_y\left\{\lambda_{\mathsf{f},w}\left(S_\mathsf{f}^{(0)}\right)\left(\nabla_x p_{\mathsf{f},w}^{(0)} + \nabla_y p_{\mathsf{f},w}^{(1)}\right)\right\} = 0. \tag{55}$$

In a standard way (see, e.g., [16]) we set :

$$p_{\mathsf{f},w}^{(1)}(x,y,t) = \sum_j \zeta_j(y)\,\frac{\partial\, p_{\mathsf{f},w}^{(0)}}{\partial x_j}(x,t) + C_1, \tag{56}$$

where $C_1 = C_1(x)$ is a constant which depends on the slow variable x, only, and ζ_j $(j = 1, ..., d)$ is the solution to the following auxiliary cell problem :

$$\begin{cases} -\Delta_y\,\zeta_j = 0 \quad \text{in } Y_\mathsf{f}; \\ \nabla_y\zeta_j\cdot\boldsymbol{\nu}_y = -\mathbf{e}_j\cdot\boldsymbol{\nu}_y \quad \text{on } \Gamma_{\mathsf{fm}} \\ y \mapsto \zeta_j(y) \quad Y-\text{periodic}. \end{cases} \tag{57}$$

Here $\boldsymbol{\nu}_y$ is a unit outer normal vector to Γ_{fm} and $\mathbf{e}_j$ is the j-th coordinate vector.

In a similar way, we obtain the representation for the corrector functions $p_{\mathsf{f},n}^{(1)} = p_{\mathsf{f},n}^{(1)}(x,y,t)$ and $\mathsf{c}_\mathsf{f}^{(1)} = \mathsf{c}_\mathsf{f}^{(1)}(x,y,t)$ by setting :

$$p_{\mathsf{f},n}^{(1)}(x,y,t) = \sum_j \zeta_j(y)\,\frac{\partial\, p_{\mathsf{f},n}^{(0)}}{\partial x_j}(x,t) + C_2 \tag{58}$$

and

$$\mathsf{c}_{\mathsf{f}}^{(1)}(x,y,t) = \sum_j \zeta_j(y)\,\frac{\partial\,\mathsf{c}_{\mathsf{f}}^{(0)}}{\partial x_j}(x,t) + C_3, \tag{59}$$

where $C_2 = C_2(x)$, $C_3 = C_3(x)$ are constants which depend on the slow variable x, only.

ε^0– equations. From the third term of the representation (34), we get the following homogenized equation involving the corrector function $p_{\mathsf{f},w}^{(2)}$:

$$\Phi_{\mathsf{f}}\frac{\partial S_{\mathsf{f}}^{(0)}}{\partial t} - K_{\mathsf{f}}\mathrm{div}_x\left\{\lambda_{\mathsf{f},w}\left(S_{\mathsf{f}}^{(0)}\right)\left(\nabla_x p_{\mathsf{f},w}^{(0)} + \nabla_y p_{\mathsf{f},w}^{(1)}\right)\right\} - K_{\mathsf{f}}\mathrm{div}_y\mathsf{Z}_{\mathsf{f},w} = \mathbb{S}_{\mathsf{f},w}^{(0)}, \tag{60}$$

where

$$\mathbb{S}_{\mathsf{f},w}^{(0)} \overset{\text{def}}{=} \mathbb{C}_{\mathsf{f},w}S_{\mathsf{f}}^{(0)}\mathsf{M}_{\mathsf{f}}^{(0)}\mathsf{c}_{\mathsf{f}}^{(0)} \tag{61}$$

and

$$\mathsf{Z}_{\mathsf{f},w} \overset{\text{def}}{=} \lambda_{\mathsf{f},w}\left(S_{\mathsf{f}}^{(0)}\right)\left(\nabla_x p_{\mathsf{f},w}^{(1)} + \nabla_y p_{\mathsf{f},w}^{(2)}\right) + \lambda'_{\mathsf{f},w}\left(S_{\mathsf{f}}^{(0)}\right)S_{\mathsf{f}}^{(1)}\left(\nabla_x p_{\mathsf{f},w}^{(0)} + \nabla_y p_{\mathsf{f},w}^{(1)}\right). \tag{62}$$

Here $\mathsf{Z}_{\mathsf{f},w}$ is an Y-periodic function in the fast variable y.

In a similar way, we obtain the second upscaled equation. It reads :

$$-\Phi_{\mathsf{f}}\frac{\partial S_{\mathsf{f}}^{(0)}}{\partial t} - K_{\mathsf{f}}\mathrm{div}_x\left\{\lambda_{\mathsf{f},n}\left(S_{\mathsf{f}}^{(0)}\right)\left(\nabla_x p_{\mathsf{f},n}^{(0)} + \nabla_y p_{\mathsf{f},n}^{(1)}\right)\right\} - K_{\mathsf{f}}\mathrm{div}_y\mathsf{Z}_{\mathsf{f},n} = \mathbb{S}_{\mathsf{f},n}^{(0)}, \tag{63}$$

where

$$\mathbb{S}_{\mathsf{f},n}^{(0)} \overset{\text{def}}{=} \mathbb{C}_{\mathsf{f},n}[1 - S_{\mathsf{f}}^{(0)}]\mathsf{M}_{\mathsf{f}}^{(0)}\mathsf{c}_{\mathsf{f}}^{(0)} \tag{64}$$

and

$$\mathsf{Z}_{\mathsf{f},n} \overset{\text{def}}{=} \lambda_{\mathsf{f},n}\left(S_{\mathsf{f}}^{(0)}\right)\left(\nabla_x p_{\mathsf{f},n}^{(1)} + \nabla_y p_{\mathsf{f},n}^{(2)}\right) + \lambda'_{\mathsf{f},n}\left(S_{\mathsf{f}}^{(0)}\right)S_{\mathsf{f},n}^{(1)}\left(\nabla_x p_{\mathsf{f},n}^{(0)} + \nabla_y p_{\mathsf{f},n}^{(1)}\right). \tag{65}$$

Here $\mathsf{Z}_{\mathsf{f},n}$ is an Y-periodic function in the fast variable y.

Finally, we turn to the concentration equation. From (51) we get :

$$\Phi_{\mathsf{f}}\frac{\partial}{\partial t}\left[S_{\mathsf{f}}^{(0)}\mathsf{c}_{\mathsf{f}}^{(0)}\right] - \mathrm{div}_x\left\{\mathcal{C}^{(0)} + \mathcal{P}_{\mathsf{c}_{\mathsf{f}}}^{(0)}\right\} - \mathrm{div}_y\left\{\mathcal{C}^{(1)} + \mathcal{P}_{\mathsf{c}_{\mathsf{f}}}^{(1)}\right\} = \mathbb{S}_{\mathsf{f},w}^{(0)},$$

Then taking into account the definitions of $\mathcal{C}^{(0)}$, $\mathcal{P}_{\mathsf{c}_{\mathsf{f}}}^{(0)}$, $\mathcal{C}^{(1)}$, and $\mathcal{P}_{\mathsf{c}_{\mathsf{f}}}^{(1)}$ (see (45), (46), (48), (49)) we have :

$$\Phi_{\mathsf{f}}\frac{\partial}{\partial t}\left[S_{\mathsf{f}}^{(0)}\mathsf{c}_{\mathsf{f}}^{(0)}\right] - \mathrm{div}_x\left\{\mathsf{c}_{\mathsf{f}}^{(0)}K_{\mathsf{f}}\lambda_{\mathsf{f},w}\left(S_{\mathsf{f}}^{(0)}\right)\left(\nabla_x p_{\mathsf{f},w}^{(0)} + \nabla_y p_{\mathsf{f},w}^{(1)}\right) + \mathcal{D}_{\mathsf{f}}\left(S_{\mathsf{f}}^{(0)}\right)\left(\nabla_x \mathsf{c}_{\mathsf{f}}^{(0)} + \nabla_y \mathsf{c}_{\mathsf{f}}^{(1)}\right)\right\} -$$

$$-\mathrm{div}_y\left\{\mathcal{C}^{(1)} + \mathcal{P}_{\mathsf{c}_{\mathsf{f}}}^{(1)}\right\} = \mathbb{S}_{\mathsf{f},w}^{(0)}. \tag{66}$$

Notice that, formally, the homogenization process is achieved at this stage. However, the equations (60), (63), and (66) still contain the corrector functions. Our next goal is to eliminate these functions from these equations and obtain the homogenized equations involving the zero order terms in the asymptotic expansions, only.

3.9 Derivation of the homogenized equations

Consider equation (60) with the function $\mathsf{Z}_{\mathsf{f},w}$ given by (62). We integrate this equation over Y_{f}. Then in a standard way (see, e.g., [16]), we have :

$$\Phi_{\mathsf{f}}\,|Y_{\mathsf{f}}|\,\frac{\partial S_{\mathsf{f}}^{(0)}}{\partial t} - \mathrm{div}_x\left\{\mathbb{K}^\star\,\lambda_{\mathsf{f},w}\big(S_{\mathsf{f}}^{(0)}\big)\nabla_x p_{\mathsf{f},w}^{(0)}\right\} + \mathcal{J}_{\mathsf{f}} = |Y_{\mathsf{f}}|\,\mathbb{S}_{\mathsf{f},w}^{(0)}\,. \tag{67}$$

Here $\mathbb{K}^\star$ is given by its entries $\mathbb{K}_{ij}^\star$ defined by :

$$\mathbb{K}_{ij}^\star \overset{\text{def}}{=} \frac{K_{\mathsf{f}}}{|Y_{\mathsf{m}}|}\int\limits_{Y_{\mathsf{f}}}\left[\nabla_y\zeta_i + \mathbf{e}_i\right]\,\left[\nabla_y\zeta_j + \mathbf{e}_j\right]\,dy, \tag{68}$$

where ζ_j $(j = 1, ..., d)$ is a solution of the cell problem (57) and

$$\mathcal{J}_{\mathsf{f}} \overset{\text{def}}{=} -K_{\mathsf{f}}\int\limits_{Y_{\mathsf{f}}}\mathrm{div}_y\left\{\lambda_{\mathsf{f},w}\big(S_{\mathsf{f}}^{(0)}\big)\,\left[\nabla_x p_{\mathsf{f},w}^{(1)} + \nabla_y p_{\mathsf{f},w}^{(2)}\right] + \lambda'_{\mathsf{f},w}\big(S_{\mathsf{f}}^{(0)}\big)S_{\mathsf{f}}^{(1)}\,\left[\nabla_x p_{\mathsf{f},w}^{(0)} + \nabla_y p_{\mathsf{f},w}^{(1)}\right]\right\}dy. \tag{69}$$

In order to rearrange $\mathcal{J}_{\mathsf{f}}$ we make use of the well known divergence theorem (see, e.g., [14]) :

$$\int\limits_{\mathcal{O}}\mathrm{div}\,w\,dx = \int\limits_{\partial\mathcal{O}} w\cdot\boldsymbol{\nu}\,d\gamma, \tag{70}$$

where $\boldsymbol{\nu}$ is the unit outer normal vector to the boundary of the domain $\mathcal{O}$ denoted by $\partial\mathcal{O}$.

Using now the relation (70), from (69), we obtain that

$$\mathcal{J}_{\mathsf{f}} = -K_{\mathsf{f}}\int\limits_{\Gamma_{\mathsf{fm}}}\left\{\lambda_{\mathsf{f},w}\big(S_{\mathsf{f}}^{(0)}\big)\,\left[\nabla_x p_{\mathsf{f},w}^{(1)} + \nabla_y p_{\mathsf{f},w}^{(2)}\right] + \lambda'_{\mathsf{f},w}\big(S_{\mathsf{f}}^{(0)}\big)S_{\mathsf{f}}^{(1)}\,\left[\nabla_x p_{\mathsf{f},w}^{(0)} + \nabla_y p_{\mathsf{f},w}^{(1)}\right]\right\}\cdot\boldsymbol{\nu}\,d\gamma, \tag{71}$$

Now we make use of (36). Namely, we have :

$$K_{\mathsf{f}}\left\{\lambda_{\mathsf{f},w}\big(S_{\mathsf{f}}^{(0)}\big)\,\left[\nabla_x p_{\mathsf{f},w}^{(1)} + \nabla_y p_{\mathsf{f},w}^{(2)}\right] + \lambda'_{\mathsf{f},w}\big(S_{\mathsf{f}}^{(0)}\big)S_{\mathsf{f}}^{(1)}\,\left[\nabla_x p_{\mathsf{f},w}^{(0)} + \nabla_y p_{\mathsf{f},w}^{(1)}\right]\right\}\cdot\boldsymbol{\nu} =$$

$$= K_{\mathsf{m}}\,\lambda_{\mathsf{m},w}\big(S_{\mathsf{m}}^{(0)}\big)\nabla_y p_{\mathsf{m},w}^{(0)}\cdot\boldsymbol{\nu}. \tag{72}$$

Then from (71), (72), we have :

$$\mathcal{J}_{\mathsf{f}} = K_{\mathsf{m}}\int\limits_{\Gamma_{\mathsf{fm}}}\left\{\lambda_{\mathsf{m},w}\big(S_{\mathsf{m}}^{(0)}\big)\nabla_y p_{\mathsf{m},w}^{(0)}\right\}\cdot\boldsymbol{\nu}_{\mathsf{m}}\,d\gamma, \tag{73}$$

where $\boldsymbol{\nu}_{\mathsf{m}}$ denotes the unit outer normal vector to the boundary Γ_{fm} of Y_{m}. Now we make use of (38). It implies the following equation in the matrix block Y_{m} :

$$\Phi_{\mathsf{m}}\,\frac{\partial S_{\mathsf{m}}^{(0)}}{\partial t} - K_{\mathsf{m}}\,\mathrm{div}_y\,\left\{\lambda_{\mathsf{m},w}(S_{\mathsf{m}}^{(0)})\nabla p_{\mathsf{m},w}^{(0)}\right\} = \mathbb{S}_{\mathsf{m},w}^{(0)}. \tag{74}$$

Then we have :

$$\mathcal{J}_{\mathsf{f}} = K_{\mathsf{m}} \int\limits_{Y_{\mathsf{m}}} \mathrm{div}_y \left\{ \lambda_{\mathsf{m},w}(S_{\mathsf{m}}^{(0)}) \nabla p_{\mathsf{m},w}^{(0)} \right\} dy = \int\limits_{Y_{\mathsf{m}}} \Phi_{\mathsf{m}} \frac{\partial S_{\mathsf{m}}^{(0)}}{\partial t} dy - \int\limits_{Y_{\mathsf{m}}} \mathbb{S}_{\mathsf{m},w}^{(0)} dy.$$

This means that equation (67) becomes :

$$\Phi_{\mathsf{f}} |Y_{\mathsf{f}}| \frac{\partial S_{\mathsf{f}}^{(0)}}{\partial t} - |Y_{\mathsf{m}}| \, \mathrm{div}_x \left\{ \mathbb{K}^{\star}(x) \, \lambda_{\mathsf{f},w}\big(S_{\mathsf{f}}^{(0)}\big) \nabla_x p_{\mathsf{f},w}^{(0)} \right\} = |Y_{\mathsf{f}}| \, \mathbb{S}_{\mathsf{f},w}^{(0)} - \int\limits_{Y_{\mathsf{m}}} \left[\Phi_{\mathsf{m}} \frac{\partial S_{\mathsf{m}}^{(0)}}{\partial t} - \mathbb{S}_{\mathsf{m},w}^{(0)} \right] dy.$$

The other homogenized equations can be obtained in a similar way.

3.10 The homogenized system

In order to formulate the first homogenized equation, we introduce the notation :

- $S \overset{\text{def}}{=} S_{\mathsf{f}}^{(0)}$, $P_w \overset{\text{def}}{=} p_{\mathsf{f},w}^{(0)}$, $P_n \overset{\text{def}}{=} p_{\mathsf{f},n}^{(0)}$, $\mathsf{c} \overset{\text{def}}{=} \mathsf{c}_{\mathsf{f}}^{(0)}$ denote the homogenized (macroscopic) wetting liquid saturation, the wetting and nonwetting liquid pressures, and concentration function, respectively.
- the averaged density of the solid reagent $\mathsf{M}_{\mathsf{f}} \overset{\text{def}}{=} \mathsf{M}_{\mathsf{f}}^{(0)}$ satisfies the following kinetic equation :

$$\begin{cases} \dfrac{\partial \mathsf{M}_{\mathsf{f}}}{\partial t} = \mathbb{S}_{\mathsf{f},\mathsf{M}} \quad \text{in } \Omega_T; \\ \mathsf{M}_{\mathsf{f}}(x,0) = 1 \quad \text{in } \Omega \end{cases} \qquad \text{with} \quad \mathbb{S}_{\mathsf{f},\mathsf{M}} \overset{\text{def}}{=} \mathbb{C}_{\mathsf{f},\mathsf{M}} \, S \, \mathsf{M}_{\mathsf{f}} \, \mathsf{c}, \tag{75}$$

- $\Phi^{\star}$ denotes the effective porosity and is given by :

$$\Phi^{\star} \overset{\text{def}}{=} \Phi_{\mathsf{f}} \frac{|Y_{\mathsf{f}}|}{|Y_{\mathsf{m}}|}, \tag{76}$$

  where $|Y_\ell|$ is the measure of the set Y_ℓ ($\ell = \mathsf{f}, \mathsf{m}$).
- $\mathbb{K}^{\star}$ is the homogenized tensor with the entries $\mathbb{K}_{ij}^{\star}$ defined by :

$$\mathbb{K}_{ij}^{\star} \overset{\text{def}}{=} \frac{K_{\mathsf{f}}}{|Y_{\mathsf{m}}|} \int\limits_{Y_{\mathsf{f}}} \left[\nabla_y \zeta_i + \mathbf{e}_i \right] \left[\nabla_y \zeta_j + \mathbf{e}_j \right] dy, \tag{77}$$

  where ζ_j is a solution of the auxiliary cell problem :

$$\begin{cases} -\Delta_y \, \zeta_j = 0 \quad \text{in } Y_{\mathsf{f}}; \\ \nabla_y \zeta_j \cdot \boldsymbol{\nu}_y = -\mathbf{e}_j \cdot \boldsymbol{\nu}_y \quad \text{on } \Gamma_{\mathsf{fm}} \\ y \mapsto \zeta_j(y) \quad Y - \text{periodic.} \end{cases} \tag{78}$$

- $\mathbb{S}_w$, $\mathbb{S}_n$ denote the source terms given by :

$$\mathbb{S}_w \overset{\text{def}}{=} \frac{|Y_{\mathsf{f}}|}{|Y_{\mathsf{m}}|} \, \mathbb{C}_{\mathsf{f},w} \, S \, \mathsf{M}_{\mathsf{f}} \, \mathsf{c} \quad \text{and} \quad \mathbb{S}_n \overset{\text{def}}{=} \frac{|Y_{\mathsf{f}}|}{|Y_{\mathsf{m}}|} \, \mathbb{C}_{\mathsf{f},n} \, [1 - S] \, \mathsf{M}_{\mathsf{f}} \, \mathsf{c}. \tag{79}$$

– $\mathbb{K}^{\mathcal{D}}(S)$ is the homogenized tensor defined by its entries :

$$\mathbb{K}^{\mathcal{D}}_{ij}(S) \overset{\text{def}}{=} \frac{\mathcal{D}(S)}{|Y_{\mathsf{m}}|} \int\limits_{Y_{\mathsf{f}}} \left[\nabla_y \zeta_i + \mathbf{e}_i\right] \left[\nabla_y \zeta_j + \mathbf{e}_j\right] dy. \tag{80}$$

Thus the homogenized system reads :

$$\begin{cases} \Phi^\star \dfrac{\partial S}{\partial t} - \operatorname{div}\left\{\mathbb{K}^\star \, \lambda_{\mathsf{f},w}(S) \nabla P_w\right\} = \mathbb{S}_w + \mathcal{Q}_w \quad \text{in } \Omega_T; \\ -\Phi^\star \dfrac{\partial S}{\partial t} - \operatorname{div}\left\{\mathbb{K}^\star \, \lambda_{\mathsf{f},n}(S) \nabla P_n\right\} = \mathbb{S}_n + \mathcal{Q}_n \quad \text{in } \Omega_T; \\ \Phi^\star \dfrac{\partial}{\partial t}[S\mathsf{c}] - \operatorname{div}\left\{\mathbb{K}^\star \, \mathsf{c}\lambda_{\mathsf{f},w}\left(S\right) \nabla P_w + \mathbb{K}^{\mathcal{D}}(S) \nabla \mathsf{c}\right\} = \mathbb{S}_w + \mathcal{Q}_c \quad \text{in } \Omega_T; \\ P_c(S) = P_n - P_w \quad \text{in } \Omega_T. \end{cases} \tag{81}$$

The flow equations in the matrix block are given by :

$$\begin{cases} \Phi_{\mathsf{m}} \dfrac{\partial s}{\partial t} - \operatorname{div}\big\{K_{\mathsf{m}} \lambda_{\mathsf{m},w}(s) \nabla p_w\big\} = \mathbb{S}_{\mathsf{m},w}; \\ -\Phi_{\mathsf{m}} \dfrac{\partial s}{\partial t} - \operatorname{div}\big\{K_{\mathsf{m}} \lambda_{\mathsf{m},n}(s) \nabla p_n\big\} = \mathbb{S}_{\mathsf{m},n}; \\ \Phi_{\mathsf{m}} \dfrac{\partial}{\partial t}\big[s\, \mathsf{c}_{\mathsf{m}}\big] - \operatorname{div}\big\{K_{\mathsf{m}}\, \mathsf{c}_{\mathsf{m}}\, \lambda_{\mathsf{m},w}(s) \nabla p_w + \mathcal{D}_{\mathsf{m}}(s) \nabla \mathsf{c}_{\mathsf{m}}\big\} = \mathbb{S}_{\mathsf{m},w}; \\ p_c(s) = p_n - p_w. \end{cases} \tag{82}$$

Here $s \overset{\text{def}}{=} S^{(0)}_{\mathsf{m}}, p_w \overset{\text{def}}{=} p^{(0)}_{\mathsf{m},w}, p_n \overset{\text{def}}{=} p^{(0)}_{\mathsf{m},n}$ and $\mathsf{c}_{\mathsf{m}} \overset{\text{def}}{=} \mathsf{c}^{(0)}_{\mathsf{m}}$ denote the local wetting liquid saturation, the wetting and nonwetting liquid pressures, and the concentration in the matrix block Y_{m}. In addition, $\mathbb{S}_{\mathsf{m},w} \overset{\text{def}}{=} \mathbb{S}^{(0)}_{\mathsf{m},w}$ and $\mathbb{S}_{\mathsf{m},n} \overset{\text{def}}{=} \mathbb{S}^{(0)}_{\mathsf{m},n}$. The averaged density of the solid reagent M_{m} satisfies in the matrix block the following kinetic equation :

$$\begin{cases} \dfrac{\partial \mathsf{M}_{\mathsf{m}}}{\partial t} = \mathbb{S}_{\mathsf{m},\mathsf{M}}; \\ \mathsf{M}_{\mathsf{m}}(x,0) = 1 \end{cases} \quad \text{with} \quad \mathbb{S}_{\mathsf{m},\mathsf{M}} \overset{\text{def}}{=} \mathbb{C}_{\mathsf{m},\mathsf{M}}\, s\, \mathsf{M}_{\mathsf{m}}\, \mathsf{c}_{\mathsf{m}}, \tag{83}$$

For any $x \in \Omega$ and $t > 0$, the matrix-fracture source terms are given by :

$$\mathcal{Q}_w \overset{\text{def}}{=} -\frac{1}{|Y_{\mathsf{m}}|} \int\limits_{Y_{\mathsf{m}}} \left[\Phi_{\mathsf{m}} \frac{\partial s}{\partial t} - \mathbb{S}_{\mathsf{m},w}\right] dy; \quad \mathcal{Q}_n \overset{\text{def}}{=} \frac{1}{|Y_{\mathsf{m}}|} \int\limits_{Y_{\mathsf{m}}} \left[\Phi_{\mathsf{m}} \frac{\partial s}{\partial t} + \mathbb{S}_{\mathsf{m},n}\right] dy \tag{84}$$

and

$$\mathcal{Q}_c \overset{\text{def}}{=} -\frac{1}{|Y_{\mathsf{m}}|} \int\limits_{Y_{\mathsf{m}}} \left[\Phi_{\mathsf{m}} \frac{\partial}{\partial t}(s\mathsf{c}_{\mathsf{m}}) - \mathbb{S}_{\mathsf{m},w}\right] dy. \tag{85}$$

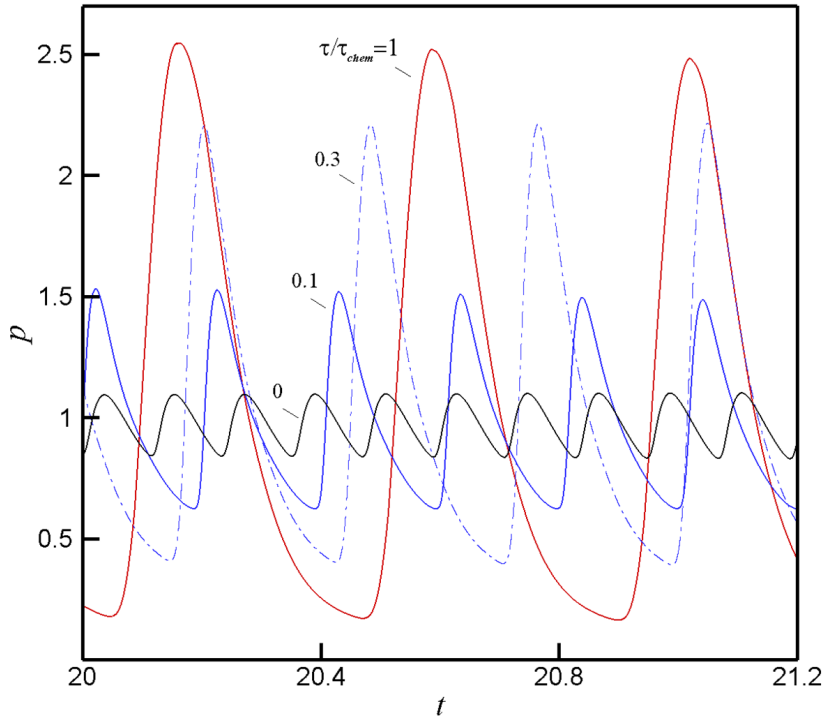

FIGURE 1: Influence of the ratio $\tau/\tau_{\rm chem}$ on the pressure oscillations : dependence of the pressure maximum on time.

4 Instability from time non-locality of the reaction rate

Let for the sake of simplicity, the concentration of the chemically active component of the skeleton and the capillary pressure vanish in the fractures. Hence, $\mathbb{S}_w = \mathbb{S}_n = 0$ in (81). Consider the $1D$ initial boundary value problem for the homogenized equations (81) in the cylinder $\Omega_T \stackrel{\rm def}{=} (0, L) \times (0, T)$, where L denotes the width of the double porosity layer. The initial state of the layer is characterized by the initial concentration of the solid reactant equal $\mathsf{M_f}(x, 0) = m_0$ and the saturation of the wetting fluid $(S(x, 0) = 1)$ with $\mathsf{c}(x, 0) = 0$. The pressure difference between left and right boundaries is supposed to be constant $\Delta p = p(0, t) - p(L, t) > 0$. The boundary condition for the concentration of the the liquid reactant at the inlet boundary is $\mathsf{c}(0, t) = \mathsf{c}_0$. Thus the problem under consideration models injection at $x = 0$ of the liquid reactant into the double porosity layer with chemically active matrix. The injection results in reaction wave propagation accompanied by production of the gas phase (represented by $\mathcal{Q}_n$ term in (81)) and two-phase flow in the layer.

Under the assumption $p_c = 0$ the matrix problem (82) introduces a time scale τ being the characteristic time of diffusion in the matrix block (in this case convection in the matrix blocks is determined by the reaction rate, which, in turn, is determined by the countercurrent diffusion). If $\tau \to 0$, the problem (82) becomes elliptic and the source terms in the homogenized equations depend only on the instant concentration c, saturation s and the distribution of the solid reactant in the matrix block. This terms vanish if $\mathsf{c} = 0$ or $s = 0$ or $\mathsf{M_m} = 0$. It is clear that the evaluation of $\mathcal{Q}_\alpha$ requires to solve the cell problem. For the sake of simplicity, let us take this local source terms in the following form

$$\mathcal{Q}_w \stackrel{\rm def}{=} \mathbb{C}_{\mathsf{f},w}\widehat{r}(t); \quad \mathcal{Q}_n \stackrel{\rm def}{=} \mathbb{C}_{\mathsf{f},n}\widehat{r}(t); \quad \mathcal{Q}_c \stackrel{\rm def}{=} \mathbb{C}_{\mathsf{f},c}\widehat{r}(t), \tag{86}$$

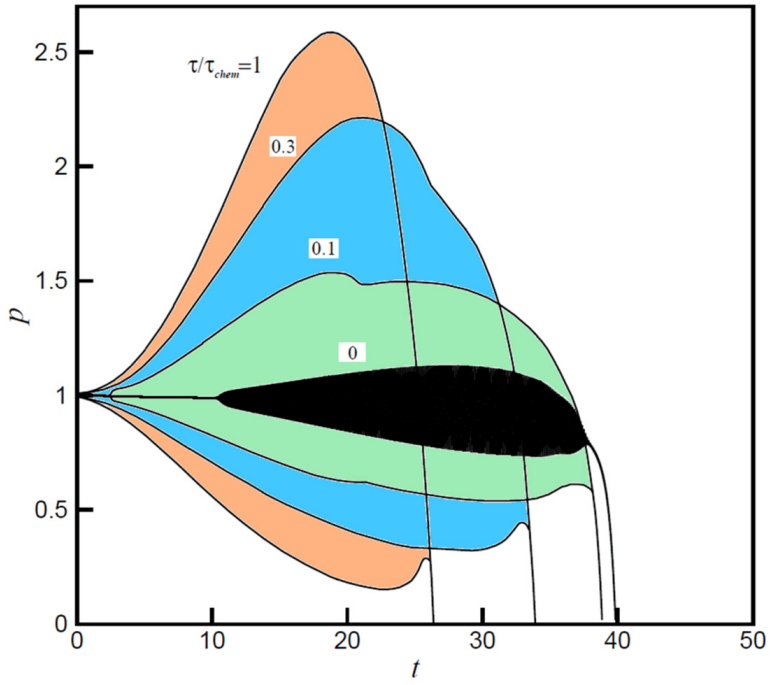

FIGURE 2: Influence of the ratio τ/τ_{chem} on the pressure oscillations : the envelope curves of pressure oscillations at the front of the reaction.

where $\widehat{r}(t) = (\tau_{chem})^{-1}\, m\, s\, \mathsf{c}$, $\tau_{chem} =$ constant. Note that the source terms (86) are local in time. In this case the homogenized equations are similar to the equations of the flow in a homogeneous medium with upscaled (macroscopic) kinetics, which depends on instantaneous values of homogenized concentrations of the reactants. However, if τ is large enough in comparison with the characteristic time scale of the problem, the time non-locality of the source terms should be taken into account, which originates from parabolicity of (82). Due to the finite diffusion time there is a time delay between the change of $\mathsf{c} = 0$ or $s = 0$ and the resulting change of homogenized reaction sources. One of the most simple non-local dependence, which models this phenomenon, is

$$\mathcal{Q}_w \stackrel{\text{def}}{=} \mathbb{C}_{\mathsf{f},w} r(t); \quad \mathcal{Q}_n \stackrel{\text{def}}{=} \mathbb{C}_{\mathsf{f},n} r(t); \quad \mathcal{Q}_c \stackrel{\text{def}}{=} \mathbb{C}_{\mathsf{f},c} r(t). \tag{87}$$

$$r(t) = \frac{1}{\tau} \int_0^t \widehat{r}(\xi)\; exp\left(\frac{\xi - t}{\tau}\right) d\xi, \quad \widehat{r} = Da \cdot \mathsf{M}_{\mathsf{m}}\, s\, c, \quad Da = t_{\text{ref}}/\tau_{\text{chem}}. \tag{88}$$

According to (88) $r \to \widehat{r}$ if $\tau \to 0$. Consider the influence of τ/τ_{chem} on the stability of the reaction wave propagation. The problem was solved numerically with dimensionless parameters of the problem $\mathsf{c}_0 = 0.5$, $\mathsf{M}_{\mathsf{m}}(x, 0) = 0.1$, and $Da = (L^2 \mu_w \Phi^\star)/(\tau_{chem} \mathbb{K} \Delta p) = 100$. The equations were approximated on uniform grid and integrated using IMPES method for different values of the parameter τ ($\tau/\tau_{\text{chem}} = 0; 0.1; 0.3; 1$).

The solutions show instability of the stationary mode of reaction front propagation. The instability manifests itself in spontaneous transition to the self-oscillating mode, in which the flow parameters in the reaction zone oscillate in time with some frequency (see, Fig. 1, where the dependence on time of the maximum pressure is presented for different τ/τ_{chem}). In Fig. 2 the envelope curves of pressure oscilla-

tions at the front of the reaction are shown. One can conclude that even small non-locality in time of the sources representing the chemical reaction stimulates the instability and transition to the self-oscillating mode.

Concluding remarks

In this paper we continue to study the non-equilibrium two-phase flow models. Our previous results are presented in the monograph [20]. We consider the two main directions of our future study : the rigorous mathematical analysis of the model (5)-(17) and a deeper numerical simulation of the initial and homogenized models. This work will be done in the forthcoming papers.

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