Generalized formulation of acoustics

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Résumé :

En 1937, Janowski et Spandöck ont mesuré la courbure des rayons sonores se propageant à incidence rasante au-dessus de matériaux absorbants. 30 ans plus tard, Cremer et Müller ont montré que cette courbure est créée par l'admittance de surface du matériau ; cependant, ils n'ont pas été en mesure de démontrer la courbure des rayons. En utilisant le tenseur énergie-impulsion, introduit en acoustique par Morse et Ingard, nous montrons que ce formalisme, emprunté à la théorie de la relativité générale, permet de dériver la courbure des rayons rasants, à condition que les coordonnées généralisées et le tenseur métrique vérifient la condition d'admittance au bord. Nous montrons que le tenseur métrique peut être défini arbitrairement, avec la seule condition que la dérivée normale du potentiel des vitesses soit nulle. L'absorption est ensuite prise en compte par les symboles de Christoffel résiduels qui définissent la dérivation normale du tenseur d'énergie-impulsion sur les bords. Une formulation généralisée de la conservation de l'énergie est alors obtenue. Ces résultats généralisent nos travaux antérieurs sur la propagation de l'énergie dans les couloirs et salles applaties.

Abstract :

In 1937, Janowski and Spandöck experimentally demonstrated the curvature of sound rays travelling at grazing incidence above absorbing materials. 30 years later, Cremer and Müller showed that this curvature was created by the surface admittance of the material ; however, they were not able to demonstrate the curvature of the rays. By making use of the stress-energy tensor, introduced in Acoustics by Morse and Ingard, we show that this formalism, borrowed from the general relativity theory, makes it possible to derive the curvature of grazing rays, provided that general coordinates are adopted so that the metric tensor adapts itself to the admittance at the boundary. We show that the metric tensor can be arbitrarily defined, with the only condition that the normal derivative of the velocity potential be null. Absorption is then taken into account by the residual Christoffel symbols that define normal derivation of the stress-energy tensor at the boundary. A generalized formulation of energy conservation is then obtained. It generalizes earlier work on energy propagation in corridors and flat rooms.

Mots clefs : general linear acoustics, generalized coordinates, absorption, conservation of energy

1 Introduction

This work is an attempt to derive a generalized formulation for the wave equation in linear acoustics, including the conservation of energy. It makes use of the conservation of the stress-energy tensor, that is, the covariance of the stress-energy tensor is null in linear acoustics, as first proven by P.M. Morse and K.U. Ingard [MI68]. At stake is a novel approach to sound field computation that naturally accounts for losses, whereas current computational methods do not.

As a matter of fact, the stress-energy tensor and its covariant conservation are the basis of general relativity. The present paper therefore investigates to which extend general relativity has application in linear acoustics. Except for its derivation by Morse and Ingard [MI68], there has been no attempt to apply this formalism to acoustics. The motivation behind this investigation was the search for a natural setting for explaining the curvature of rays above absorbing surfaces experimentally proven by Janowsky and Spandöck in the 30s [JS37], and confirmed by Cremer and Müller [CM78].

2 The generalized wave equation

We consider a 4-dimensional time-space with its metric tensor g_{ij} and the volume element $dV = \sqrt{|g|}dx^0 \dots dx^3$, where $g = \det g_{ij}$ [Lin 5]. The infinitesimal distance element is given by :

$$ds^2 = g_{ij} dx^i dx^j \tag{1}$$

and the generalized wave equation is :

$$\Box\Phi = \nabla_i g^{ij} \nabla_j \Phi = 0 \tag{2}$$

where Φ is the velocity potential and g^{ij} the inverse matrix of g_{ij}. Note that ∇_i is the covariant derivation with respect to x^i, which differs from the usual partial derivation ∂_i in a way that depends on the tensor rank. For example, for a function Φ :

$$\nabla_j \Phi = \partial_j \Phi = \Phi_j$$

but $\nabla_j \Phi_i = \partial_j \Phi_i - \Gamma^k_{ji} \Phi_k$, where the Γ^k_{ji} are the Christoffel symbols linked to the derivatives of the elements of the metric tensor g^{ij} :

$$\Gamma^k_{ji} = \frac{1}{2} g^{kl} \left(\partial_j g_{il} + \partial_i g_{lj} - \partial_l g_{ji} \right) \tag{3}$$

Note that, just like ordinary differentiations, covariant derivations commute ; and that, by construction, covariant derivatives commute with the elements of the metric tensor [Lin 5]. In other words, the contravariant derivation ∇^i is equal to :

$$\nabla^i = g^{ij} \nabla_j = \nabla_j g^{ij}$$

3 Volume deformation

In general, the velocity potential Φ is a complex function. So one can consider the product $\Phi^* \Box \Phi$, where Φ^* is the complex conjugate of Φ. Differentiation rules lead to :

$$\Phi^* \Box \Phi = \Phi^* \nabla_i g^{ij} \nabla_j \Phi = \nabla_i g^{ij} \left[\Phi^* \nabla_j \Phi \right] - \left[\nabla_i \Phi^* \right] g^{ij} \nabla_j \Phi = 0 \tag{4}$$

that is, to :

$$\nabla_i g^{ij} \left[\Phi^* \nabla_j \Phi\right] = \left[\nabla_i \Phi^*\right] g^{ij} \nabla_j \Phi \tag{5}$$

As the right member of the preceding equation is real, considering only the real part leads to :

$$\nabla^j \frac{\Phi^* \nabla_j \Phi + \Phi \nabla_j \Phi^*}{2} = \nabla_i \Phi^* g^{ij} \nabla_j \Phi = 2L \tag{6}$$

where L is the Lagrangian. It should be noticed that $\frac{\Phi^* \nabla_j \Phi + \Phi \nabla_j \Phi^*}{2}$ can be rewritten as :

$$\frac{\Phi^* \nabla_j \Phi + \Phi \nabla_j \Phi^*}{2} = \nabla_j \left(\frac{1}{2}|\Phi|^2\right) \tag{7}$$

that is, as the 4-gradient of a real function. As a consequence, $\frac{\Phi^* \nabla_j \Phi + \Phi \nabla_j \Phi^*}{2}$ can be assimilated to a 4-velocity $-\mathfrak{V}_j$. Thus the real part of Eq. (5) simply reduces to :

$$\nabla^i \mathfrak{V}_i = -2L \tag{8}$$

meaning that the Lagrangian L amounts to a volume deformation (divergence of a velocity).

4 Conservation of the stress-energy tensor

We now consider the product $\nabla_k \Phi^* \Box \Phi$. Once more, differentiation rules lead to :

$$\begin{aligned}
\nabla_k \Phi^* \Box \Phi = \nabla_k \Phi^* \nabla_i g^{ij} \nabla_j \Phi &= \nabla_i g^{ij} \left[\nabla_k \Phi^* \nabla_j \Phi\right] - \left[\nabla_i \nabla_k \Phi^*\right] g^{ij} \nabla_j \Phi \\
&= \nabla_i g^{ij} \left[\nabla_k \Phi^* \nabla_j \Phi\right] - \left[\nabla_k \nabla_i \Phi^*\right] g^{ij} \nabla_j \Phi \\
&= \nabla^j \left[\nabla_k \Phi^* \nabla_j \Phi\right] - \left[\nabla_k \nabla_i \Phi^*\right] g^{ij} \nabla_j \Phi = 0
\end{aligned}$$

As i and j are mute indices, considering only the real part of the preceding equation leads to :

$$\nabla^j \left(\nabla_j \Phi^* \nabla_k \Phi + \nabla_j \Phi \nabla_k \Phi^*\right) = \nabla_k \left(\nabla_i \Phi^* g^{ij} \nabla_j \Phi\right) \tag{9}$$

that is, to :

$$\nabla^i T_{ij} = 0 \tag{10}$$

where T_{ij} is the *symmetrical* stress-energy tensor, defined by :

$$T_{ij} = \frac{\nabla_i \Phi^* \nabla_j \Phi + \nabla_i \Phi \nabla_j \Phi^*}{2} - \frac{1}{2} g_{ij} \left(\nabla_i \Phi^* g^{ij} \nabla_j \Phi\right) \tag{11}$$

It is easy to recognise that this equation corresponds to the contravariant conservation of the stress-energy tensor.

5 Solving the generalized wave equation

Let us now consider a 4-dimensional manifold V, with border $\partial V = S$, metric tensor g_{ij} and volume element $dV = \sqrt{|g|}dx^0 \dots dx^3$. We consider boundary conditions of the admittance type on S :

$$\nabla_i \Phi \beta^i = 0 \tag{12}$$

where $\beta^i = \sigma^i + i\zeta^i$ is the complex 4-admittance. Solving the wave equation in V with boundary condition (12) on S amounts to finding the extremum of the Lagrangian, that is, the extremum of the volume integral $\mathfrak{T} = \int_V TdV$ of the volume deformation $2L$ given by Eq. (8). Using Eq. (6), we thus obtain :

$$2\mathfrak{L} = 2\int_V LdV = \frac{1}{2}\int_V \nabla_i g^{ij}\left[\Phi^*\nabla_j\Phi + \Phi\nabla_j\Phi^*\right]\sqrt{|g|}dx^0\dots dx^n \tag{13}$$

Using the identity $\nabla_i X^i = \frac{1}{\sqrt{|g|}}\partial_i(\sqrt{|g|}X^i)$ [Lin 5] and applying Stokes theorem, the last expression leads to :

$$2\int_V LdV = \frac{1}{2}\int_S \left[\Phi^*\nabla_j\Phi + \Phi\nabla_j\Phi^*\right] g^{ji}n_i dS \tag{14}$$

Choosing the local coordinates on the boundary so that n_i is the *outgoing* 4-vector, and that $g^{ji}n_i = \sigma^j$ leads to :

$$\begin{aligned}\left[\Phi^*\nabla_j\Phi + \nabla_j\Phi^*\Phi\right] g^{ji}n_i &= \left[\Phi^*\nabla_j\Phi + \nabla_j\Phi^*\Phi\right]\sigma^j \\ &= \left[\Phi^*\nabla_j\Phi\beta^j + \nabla_j\Phi^*\beta^{*j}\Phi\right] - \left[i\zeta^j\Phi^*\nabla_j\Phi - i\zeta^j\nabla_j\Phi^*\Phi\right] \\ &= 0 - i\zeta^j\left[\Phi^*\nabla_j\Phi - \nabla_j\Phi^*\Phi\right] = 2\zeta^j\Im\left[\Phi^*\nabla_j\Phi\right]\end{aligned} \tag{15}$$

Thus :

$$2\mathfrak{L} = 2\int_V TdV = \int_S \zeta^j\Im\left[\Phi^*\nabla_j\Phi\right]dS \tag{16}$$

It should be noted that the metric tensor g^{ij} is *normalized* by $|g| = c^2$, but its elements are not bounded. Similarly, the β^i are not bounded.

6 Example : Propagation above constant absorbing plane

We now give an example of a metric tensor g_{ij} that satisfies the admittance condition (12) on the boundary, with real admittance : the case of sound propagation above an absorbing plane $x^1 = 0$.

Boundary condition (12) then reduces to $\nabla_0\Phi\beta^0 + \nabla_1\Phi\beta^1 = 0$, with $\frac{\beta^0}{\beta^1} = -\sigma$, the constant real admittance on the absorbing plane. As the outgoing 4-vector is $n_i = (0, -1, 0, 0)$, the local metric must satisfy $g^{j1}n_1 = \sigma^j$, that is, $g^{10} = g^{01}$ is non-zero. In other words, the metric tensor and its inverse are no longer diagonal, and must be written as :

$$g^{ij} = \begin{pmatrix} -c & b & 0 & 0 \\ b & a & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ g_{ij} = \begin{pmatrix} -a & b & 0 & 0 \\ b & c & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{17}$$

where a, b, and c only depend on the coordinate x^1 and not on the two other space coordinates. The boundary condition is then given $\nabla_0\Phi b + \nabla_1\Phi a = 0$, with $\frac{b}{a} = -\sigma$, the real admittance on the absorbing plane. Normalizing a, b, and c by $ac + b^2 = 1$, it is then obvious that $g = \det(g^{ij}) = -1$, $a > 0$, and $b \leq 0$.

Lagrangian Simple calculation shows that the Lagrangian L is given by :

$$L = \frac{1}{2}\left[-c|\Phi_0|^2 + b(\Phi_0^*\Phi_1 + \Phi_0\Phi_1^*) + a|\Phi_1|^2 + |\Phi_2|^2 + |\Phi_3|^2\right] \tag{18}$$

Christoffel symbols The Christoffel symbols are all equal to 0, except for :

$$\Gamma^0_{00} = \Gamma^1_{00} = -\Gamma^1_{01} = \frac{1}{2}ba_1, \Gamma^0_{01} = \frac{1}{2}ca_1 \tag{19}$$

$$\Gamma^0_{10} = -\Gamma^1_{11} = \frac{1}{2}ca_1, \Gamma^1_{10} = -\frac{1}{2}ba_1, \Gamma^0_{11} = -cb_1 + \frac{1}{2}bc_1 \tag{20}$$

Stress-energy tensor The conservation of the stress-energy tensor takes a simpler form for T^i_j than for T_{ij}, event though it is no longer symmetric. Indeed, Eq (10) can be written as :

$$\nabla^i T_{ij} = g^{ik}\nabla_k T_{ij} = \nabla_i T^i_j = \partial_i T^i_j + \Gamma^i_{in} T^n_j - \Gamma^n_{ij} T^i_n = 0 \tag{21}$$

We then obtain :

$$\partial_0 T^0_0 + \partial_1 T^1_0 + \partial_2 T^2_0 + \partial_3 T^3_0 - \frac{1}{2}ba_1 T^0_0 - \frac{1}{2}ca_1 T^1_0 + \frac{1}{2}ba_1 T^0_1 + \frac{1}{2}ba_1 T^1_1 = 0 \tag{22}$$

$$\partial_0 T^0_1 + \partial_1 T^1_1 + \partial_2 T^2_1 + \partial_3 T^3_1 - \frac{1}{2}ca_1 T^0_0 + (cb_1 - \frac{1}{2}bc_1)T^1_0 + \frac{1}{2}ba_1 T^0_1 + \frac{1}{2}ca_1 T^1_1 = 0 \tag{23}$$

$$\partial_0 T^0_2 + \partial_1 T^1_2 + \partial_2 T^2_2 + \partial_3 T^3_2 = 0 \tag{24}$$

$$\partial_0 T^0_3 + \partial_1 T^1_3 + \partial_2 T^2_3 + \partial_3 T^3_3 = 0 \tag{25}$$

Ray equation Sound rays simply follow the equation $dx^i g_{ij} dx^j = 0$. With the notations :

$$x' = \frac{dx^1}{dx^0},\ y' = \frac{dx^2}{dx^0},\ z' = \frac{dx^3}{dx^0}$$

it reduces to Monge equation :

$$2bx' + c(x')^2 + (y')^2 + (z')^2 = a$$

Ray curvature can then be computed form the generalized acceleration equation :

$$\frac{dv^i}{d\tau} = -\Gamma^i_{kl} v^k v^l$$

where τ is the proper time defined by $d\tau^2 = -ds^2 = -g_{ij}dx^i dx^j$ and the v^i are defined by :

$$v^i = \frac{dx^i}{d\tau}$$

We can then write :

$$\begin{aligned}\frac{dv^i}{d\tau} &= \frac{d}{d\tau}\frac{dx^i}{d\tau} = -\Gamma^i_{kl}v^k v^l \\ &= \frac{d}{d\tau}\left(\frac{dx^i}{dx^0}\frac{dx^0}{d\tau}\right) = \frac{d^2x^i}{(dx^0)^2}\left(\frac{dx^0}{d\tau}\right)^2 + \frac{dx^i}{dx^0}\frac{d^2x^0}{(d\tau)^2} \\ &= \frac{d^2x^i}{(dx^0)^2}(v^0)^2 - \frac{dx^i}{dx^0}\Gamma^0_{kl}v^k v^l\end{aligned}$$

Thus :

$$(x^i)'' = \frac{d^2x^i}{(dx^0)^2} = \left(\frac{dx^i}{dx^0}\Gamma^0_{kl} - \Gamma^i_{kl}\right)\frac{v^k}{v^0}\frac{v^l}{v^0} \tag{26}$$

We then obtain the system :

$$\begin{aligned}x'' &= x'\left[\Gamma^0_{00} + 2\Gamma^0_{01}x' + \Gamma^0_{11}(x')^2\right] - \left[\Gamma^1_{00} + 2\Gamma^1_{01}x' + \Gamma^1_{11}(x')^2\right] \\ &= \frac{1}{2}\left[ba_1 + 3ba_1x' + 3ca_1(x')^2 - (2cb_1 - bc_1)(x')^3\right] \qquad (27)\\ y'' &= y'\left[\Gamma^0_{00} + 2\Gamma^0_{01}x' + \Gamma^0_{11}(x')^2\right] \\ &= \frac{1}{2}\left[ba_1 + 2ca_1x' + -(2cb_1 - bc_1)(x')^2\right]y' \qquad (28)\\ z'' &= z'\left[\Gamma^0_{00} + 2\Gamma^0_{01}x' + \Gamma^0_{11}(x')^2\right] \\ &= \frac{1}{2}\left[ba_1 + 2ca_1x' + -(2cb_1 - bc_1)(x')^2\right]z' \qquad (29)\end{aligned}$$

Since a must increase away from the boundary and b is negative, ba_1 is negative and a ray parallel to the absorbing plane ($x' = 0$) is curved down toward the plane, as experimentally observed by Janowsky and Spandöck [JS37]. On the other hand, rays with normal incidence ($y' = z' = 0$) are not deviated and remain normal to the surface.

7 Conclusion

Besides deriving the wave equation and the corresponding energy conservation in generalized coordinates, the present paper has proved the possibility to construct metric tensors g^{ij} that satisfy the boundary condition $g^{ji}n_i = \sigma^i$ on the border S. However, the initial hypothesis that an analogue to Einstein equation holds in acoustics cannot be maintained. Even though the corresponding Ricci curvature tensor R^{ij} satisfies the relation

$$\nabla_i\left[R^{ij} - \frac{1}{2}g^{ij}R\right] = 0$$

with R the scalar curvature, the stress-energy tensor T^{ij} cannot be proportional to the Einstein tensor $G^{ij} = R^{ij} - \frac{1}{2}g^{ij}R$, as the metric tensor must be time invariant, whereas the stress-energy tensor is decaying with time. The solution of the generalized wave equation must therefore be numerically computed.

Beyond the falsification of the initial hypothesis, some new results have been obtained. Firstly, a generalized equation for energy conservation has been derived, that can now be numerically solved along the line of our previous papers [DPTP17, DPTP18]. Secondly, the thickness of "adaptation layer" between the surface admittance and the free space above is a free parameter in the theory ; it can thus be adjusted to the case at hand, for example to a quarter wavelength as usually hypothesized in room acoustics.

Thirdly, ray curvature above finite impedance surfaces has been proved, and a proper explanation to the experimental finding of Janowsky and Spandöck [JS37] can now be given, or to the computational artifice, amounting to multiple reflections of the rays on the boundaries, currently used in Boudary Element Methods (BEM) to compute propagation above flat lossy boundaries of varying admittance [Ras82]. Such results provide a validity check of the present generalized formulation for acoustics.

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