# ANALYTICAL AND NUMERICAL EVALUATION OF MIXED MODE CRACK PROBLEM FOR THREE-DIMENSIONAL MEDIUM USING AN INVARIANT INTEGRAL

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## Résumé :

*La durabilité des structures est confrontée à des problèmes de fissurations sous chargements mécaniques complexes associées à des variations climatiques sévères. L'objectif principal de ce travail est de présenter une nouvelle modélisation mathématique pour les problèmes de fissurations tridimensionnelle sous chargement en mode mixte. Le modèle repose sur nouvelle intégrale, intitulée M3D, dont l'implémentation dans un logiciel par élément finis permet d'obtenir le taux de restitution d'énergie ou la ténacité associée. L'intégrale M3D est développée grâce à la méthode du champ $\theta$. Les résultats montrent comment évolue la résistance à la fissuration en fonction de différents taux de mixité. Pour cela l'éprouvette appelée MMCG est utilisée. Cette éprouvette permet d'observer une plage importante de stabilité le long de la fissure en mode mixte.*
*La propriété de l'indépendance du domaine d'intégration est vérifiée. Le taux de restitution d'énergie est calculé le long du front de la fissure. Le découplage des modes de rupture est réalisé pour plusieurs configurations. Plusieurs comparaisons sont réalisées et une attention particulière est accordée aux effets du cisaillement hors plan (mode III).*

## Abstract :

*The work proposed is a contribution to a better understanding the cracking mechanisms under mixed mode loading by analytical and numerical modelling in fracture mechanics in the three-dimensional medium. The fracture usually occurs as a result of a multitude of combined mechanical loads that are accompanied by multiple secondary disruptions, such as climatic variations over time. Predictive*

*mathematical models are needed to quantify fracture parameters in the three-dimensional medium, and then predict the behaviour of cracking and assess the risk of failure. The three-dimensional modelling of the fracture is important because we do not have to perform destructive tests to conduct different investigations. The assumptions of plane strain or plane stress considered in two-dimensional case are not always valid. The third dimension in the mathematical model allows a better understanding of all the phenomena highlighted during the cracking, In particular in massive structures where cracking occurs inside the material.*
*In this work, a mathematical formulation based on an energy approach is developed for the study of mixed mode crack problem in three-dimensional medium. Thanks to Noether's theorem, it proposes a generalization of the surface and volume integrals in order to compute the energy release rate and its distribution along the crack front. This generalization will allow the fracture modes separation due to the introduction of a virtual displacement field.*
*A numerical validation, in terms of energy release rate, is carried out on a Mixed Mode crack Growth MMCG specimen under mixed-mode loading for different thickness. The finite element numerical computation are conducted to validate the invariance property of the proposed new integral, as well as comparisons with two-dimensional case. The invariance property of the integration path around the crack front is verified for the each fracture modes, as well as the influence of the fracture modes evaluated according to the thickness exploration value. The fracture mode separation is extended to evaluate out of plan complex loads; mode III, mixed mode I + II + III and mixed mode I + III.*

**Key words : Crack problem ; Energy release rate ; Mixed mode ; invariant integral ; Three-dimensional medium**

# 1 Introduction

In a previous work, we proposed a new three-dimensional independent integral entitled J3D and Gθ (3D) which make possible studying the fracture process by considering three-dimensional effects [1]. In this paper, this is a question of extended this integral to mixed-mode loading case in three-dimensional medium and takes into account different mixed-mode ratios. This integral will be entitled M3D integral. This generalization will allow us mixed-mode separation for complex geometries and loadings, including tensile, shear and torsion fracture modes. The M3D integral is presented, and then implemented using the virtual local displacement field, to be able to separate the fracture mixed modes. The numerical computation is performed for an isotropic elastic material for static crack. This new integral is validated numerically through the verification of the invariance of the integration domain property for several thicknesses. Then, an additional verification consists to compare the results with the two-dimensional case through the two assumptions of plane stresses and plane strain is carried out for two different thicknesses (1 and 25mm). Different results of the non-dependence of the integration domain property is presented. Afterwards, the evaluation of the energy release rate Gθ GI and GII along crack front are obtained. Thanks to the mixed mode separation it become possible to obtain GIII for out of plan complex loads. In the third part, the fracture mode separation is extended to evaluate out of plan complex loads; mode III, mixed mode I + II + III and mixed mode I + III. Finally, the limits of three-dimensional modeling using the current virtual displacement field are evoked to put into perspective the need to develop a local approach adapted to the case of a three-dimensional medium.

# 2 Analytical formulation

For plan problem and for static crack Rice (1968) has defined A path independent integral which allows to computes energy release rate around the crack tip [1]. J-integral takes the following notation:

$$J^{2D} = \int_{\Gamma} \left(W.n_1 - (\sigma_{ij}.n_j.u_{i,1})\right).d\Gamma \tag{1}$$

Where $W$ denotes the strain energy density, $\Gamma$ is arbitrary curvilinear contour oriented by its normal vector, $u_i$ is the displacement component and $\sigma_{ij}$ is the stress component.

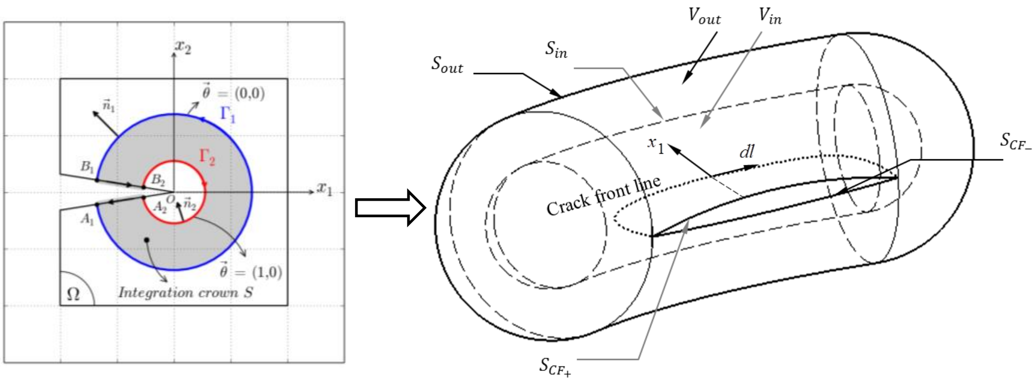


Fig. 1: Description integral domain for 2D and 3D crack

To take into account in global formalism the most general case which the crack front line is arbitrary, we consider some volume and surfaces integration domains (figure 1).

The development of the three-dimensional M-integral concept is based on a Noether's theorem [2]. The M3D-integral formulation is based on energetic approach using consideration described in figure 1 [3]. It expressed as follow.

$$\begin{aligned} M^{3D} =& \frac{1}{2}\int_{S_{\Gamma_1}} \left( \left(\sigma^v_{ij}.u_{i,j} + \sigma^u_{ij}.v_{i,j}\right).\delta_{kj} - \left(\sigma^v_{ij}.u_{i,k} + \sigma^u_{ij}.v_{i,k}\right)\right).n_j.dS \\ &+ \frac{1}{2}\int_{V_{\Gamma_1}} \left( \left(\sigma^v_{ij}.\left(\varepsilon^u_{ij}\right)_{,k} + \sigma^u_{ij}.\left(\varepsilon^v_{ij}\right)_{,k}\right) - \left(\left(\sigma^v_{ij}.\varepsilon^u_{ij}\right)_{,k} + \left(\sigma^u_{ij}.\varepsilon^v_{ij}\right)_{,k}\right)\right).dV \\ &- \frac{1}{2}\int_{S_{CF}} \left(\sigma^v_{ij}.u_{i,k} + \sigma^u_{ij}.v_{i,k}\right).n_j.dS \end{aligned} \tag{2}$$

To implement this integral in a finite element software, it is easier to consider a volume domain integral.

$$\begin{aligned} M3D = M^{3D}_{\theta} =& \frac{1}{2}\int_{V} P_{kj}.\theta_{k,j}.dV + \frac{1}{2}\int_{V_{\Gamma_2}} \left( \left(\sigma^v_{ij}.\left(\varepsilon^u_{ij}\right)_{,k} + \sigma^u_{ij}.\left(\varepsilon^v_{ij}\right)_{,k}\right) - \left(\left(\sigma^v_{ij}.\varepsilon^u_{ij}\right)_{,k} + \left(\sigma^u_{ij}.\varepsilon^v_{ij}\right)_{,k}\right)\right).\theta_k.dV \\ &- \frac{1}{2}\int_{S_{CF}} \left(\sigma^v_{ij}.u_{i,k} + \sigma^u_{ij}.v_{i,k}\right).n_j.\theta_k.dS \end{aligned} \tag{3}$$

For static crack M3D can be expressed as follow:

$$M3D = \frac{1}{2}\int_{V} \left(\sigma^u_{ij}.v_{i,k} - \sigma^v_{ij,k}.u_i\right).\theta_{k,j}.dV \tag{4}$$

# 3 Numerical validation

The numerical implementation is based on a Mixed Mode Crack Growth (MMCG) specimen loaded in mixed mode. The MMCG specimen was developed by Moutou Pitti et al [4,5]. The geometry of the MMCG specimen has been optimized by using a finite element computation. This specimen is adapted to obtain a stable crack growth rate during propagation for mixed mode. The numerical implementation is based on Cast3m finite element software.

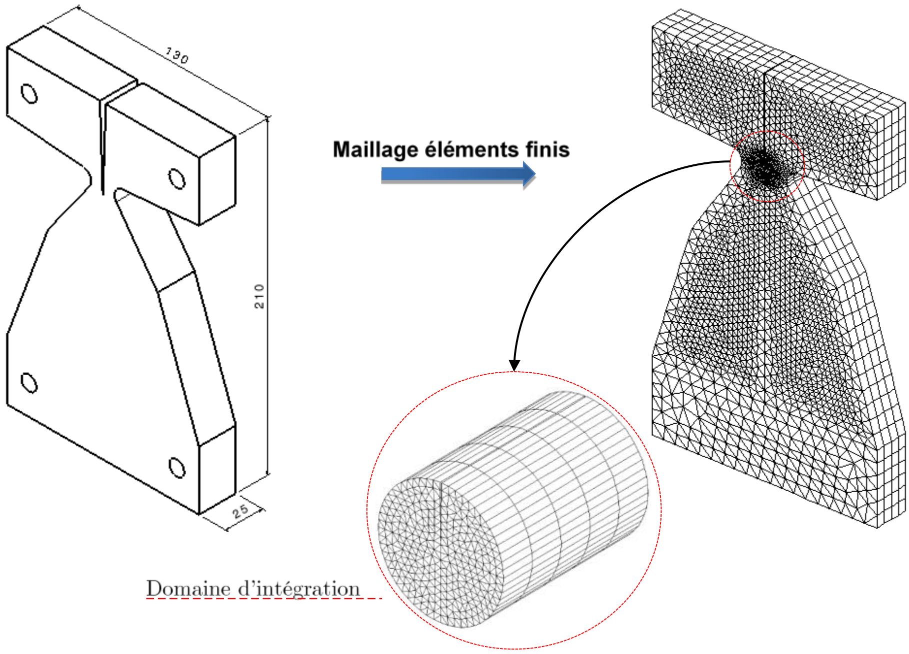


Fig. 2: MMCG specimen

The finite element computation is realized for an elastic isotropic behavior. In the follows, the results of numerical study are exposed. In order to observe the effect of thicknesses on the MMCG specimen we plot the evolution of the energy release rate as function of the crack front line. The description of θ field around the crack front line is shown in Fig. 3. The θ field is equal to zero on outside surface, and 1 on inside surface.

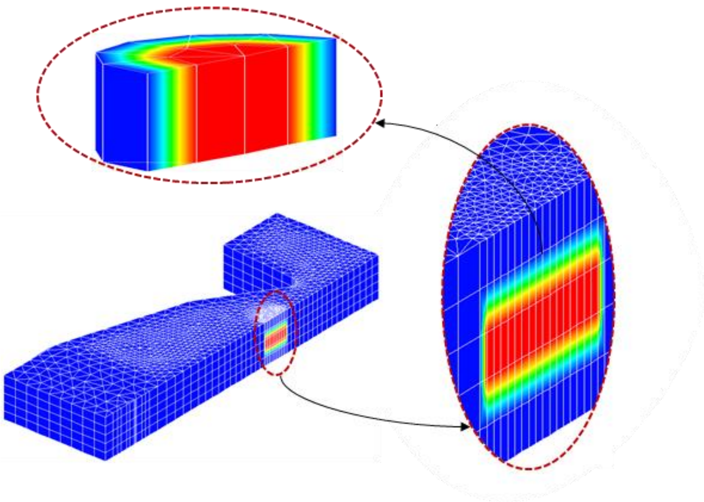

Fig. 3: Description of $\theta$ field around the crack front line

The external load is 1500N applied on the MMCG specimen. The goal of the first application is to validate the non-dependence of the integration domain by varying the size of the integration domain through its radius *R*ext used to calculate energy release rate distribution (GI, GII and GIII). The integration domain size has been parameterized with *R*ext values lying between 1 mm and 25 mm

Fig. 4 shows the variations in energy release rate vs. integration domain size. Numerical results validate the non-dependence of the integration domain. Note that for *R*ext=1mm, results present some artefacts because the mechanical fields present some singularity around the crack front.

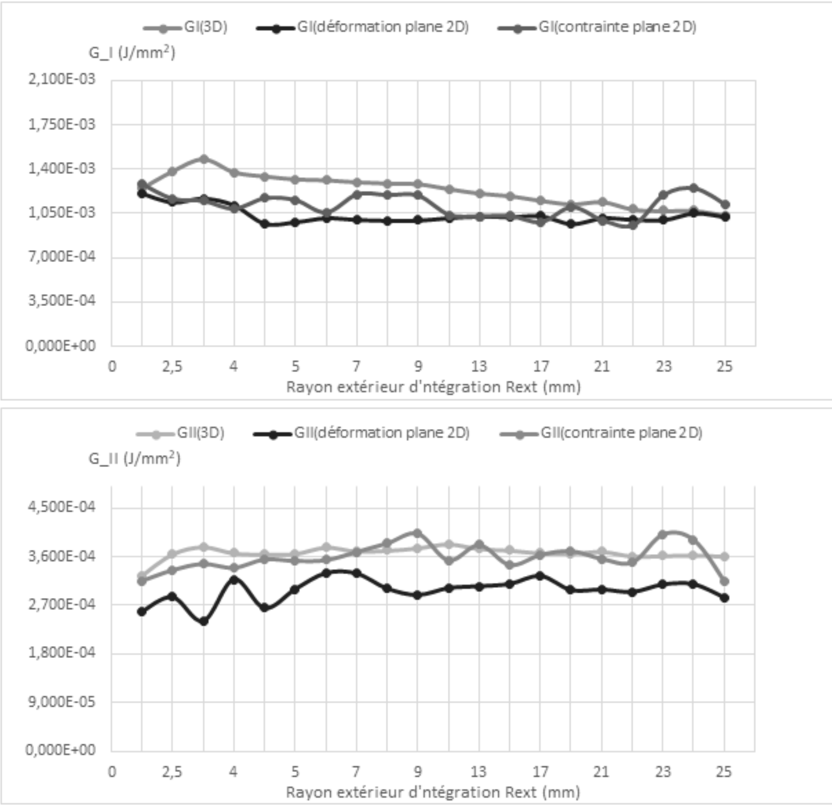


Fig. 4: Energy release rate distribution Vs integration domain around the crack front line

We observe the ability of the M3D integral to separate the fracture modes.

The second validation deals with a comparison of different thickness. In this case, two thicknesses are compared 25mm and 1mm Fig. 5, and the evaluation of the energy release rate along the crack front line are observed for each one, Fig. 6.

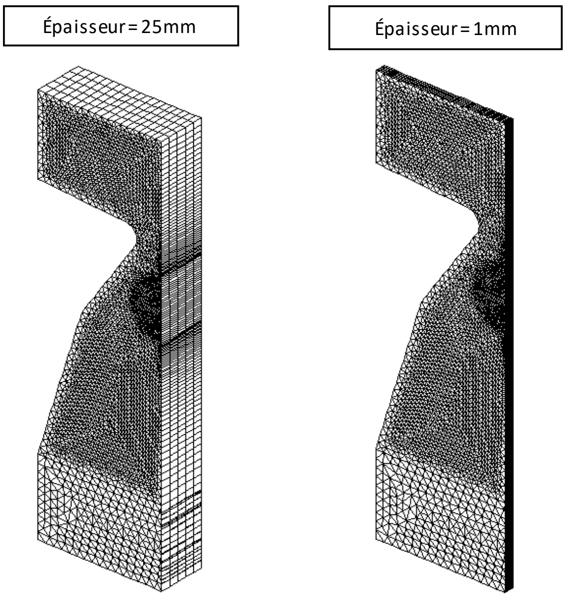


Fig. 5: Impact of the sample thickness

In this application, the crack front line is always a straight line of finite elements. For each plane cylinder, the energy release rate is calculated at the gravity center projected onto the crack front line. For a discretization of the sample stiffness with $n\omega$=10 elements, Fig. 6 indicates the energy release rate distribution for each mode GI, GII and GIII along the crack front line vs. thickness.

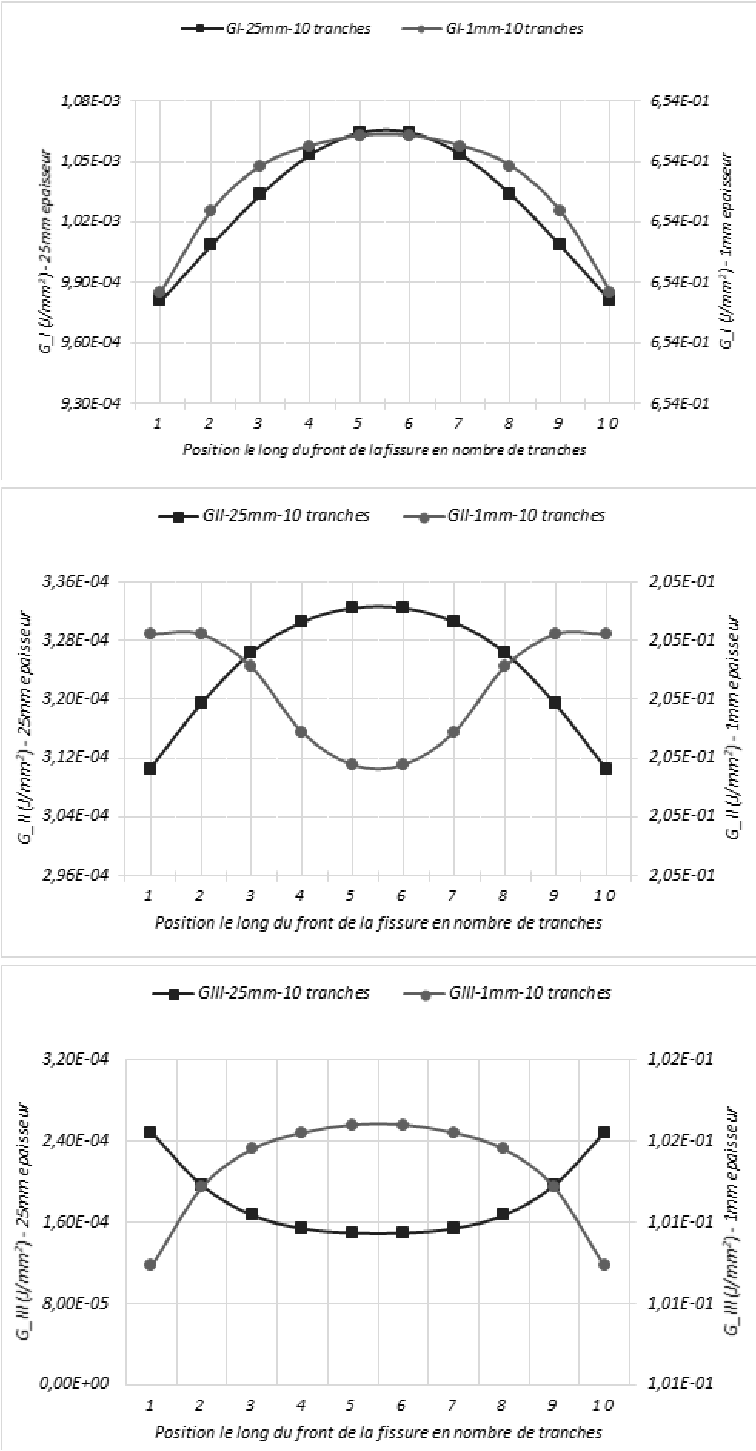


Fig. 6: Energy release rate distribution along the crack front line

This result show us in that even in the case of pure mode III loading, the Poisson effect induce systematically opening and shear mode.

# 4 Conclusions and outlooks

This paper deals with a new formulation of the mixed mode integral for the study of fracture process in element by taking into account three-dimensional effects. Non-path dependence is proven thanks to the use of an analytical formulation.
One promising perspective consists of completing this study by a theoretical approach proposing a three-dimensional vision of the local mechanical fields in the crack front vicinity.

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